Int. J. of Applied Mechanics and Engineering, 2006, vol.11, No.1, pp.179-194

TIME-DEPENDENT PIEZOELECTRIC FRACTURE BEHAVIOR OF CONDUCTING CRACKS AND ELECTRODES

B.L. WANG* and Y.-W. MAI
School of Aerospace, Mechanical and Mechatronic Engineering
The University of Sydney
Sydney, NSW 2006, AUSTRALIA
e-mail: wangbl2001@hotmail.com

This paper investigates the fracture behavior of a piezoelectric material subjected to transient electro-mechanical loads. The piezoelectric medium contains a straight-line crack, which is parallel to its poling direction. The Fourier transform technique is used to reduce the problem to the solution of singular integral equations in Laplace transform plane. The Laplace inversion yields the results in the time domain. Some useful results are obtained. Strong coupling between stress and electric field near crack tips has been found.

Key words: piezoelectric materials, dynamic fracture, crack.

1. Introduction

Many piezoelectric devices may experience transient loads. For example, devices such as phase change transducers and pulse generators for igniters and high voltage transformers are almost routinely subjected to very large voltages over very short intervals of time (Sosa and Khutoryansky, 1999; 2001). Therefore, many authors have studied the dynamic fracture of piezoelectric materials (Ueda, 2003; Nishioka *et al*., 2003; Jin *et al*., 2003; Kwon and Lee, 2003; Li and Tang, 2003; He, 2002; Ricci *et al*., 2003). The above works are limited to the insulating crack problem. Clearly, there is a need to investigate the conducting cracks in piezoelectric materials under a transient electromechanical impact. Motivated by this consideration, this paper investigates a piezoelectric strip with an electrically conducting crack under an in-plane electro-mechanical impact. Laplace and Fourier transforms are used to reduce the problem to the solution of singular integral equations. Numerical calculations are carried out and the results of the time dependent crack tip field are shown graphically to illustrate the effect of the electric fields applied.

2. Description of the problem

Referring to Fig.1, we consider a piezoelectric strip of thickness $(h_1 + h_2)$ containing a crack of length $2a$. The coordinates x and z coincide with x_1 and x_2, respectively. In what follows, u_1 and u_2 will denote the displacement components in x_1 and x_2 directions, ϕ will denote the electric potential, and t will denote the time.

The in-plane deformation is considered such that u_1, u_2 and ϕ are functions of x_1 and x_2 only. Constitutive equations for a piezoelectric material poled along the x_1-axis are

* To whom correspondence should be addressed

$$\begin{Bmatrix}\sigma_{22}\\ \sigma_{11}\\ \sigma_{12}\\ D_2\\ D_1\end{Bmatrix}=\begin{bmatrix}c_{11} & c_{13} & 0 & 0 & -e_{31}\\ c_{13} & c_{33} & 0 & 0 & -e_{33}\\ 0 & 0 & c_{44} & -e_{15} & 0\\ 0 & 0 & e_{15} & \in_{11} & 0\\ e_{31} & e_{33} & 0 & 0 & \in_{33}\end{bmatrix}\begin{Bmatrix}u_{2,2}\\ u_{1,1}\\ u_{1,2}+u_{2,1}\\ E_2\\ E_1\end{Bmatrix} \tag{2.1}$$

where c_{11}, c_{13}, c_{33} and c_{44} are elastic constants; e_{31}, e_{33} and e_{15} are piezoelectric constants; and $\in_{11}$ and $\in_{33}$ stand for dielectric permittivities; ρ is density; t is time variable; σ_{12}, σ_{22} and σ_{11} are stress components; D_1 and D_2 are electric displacements.

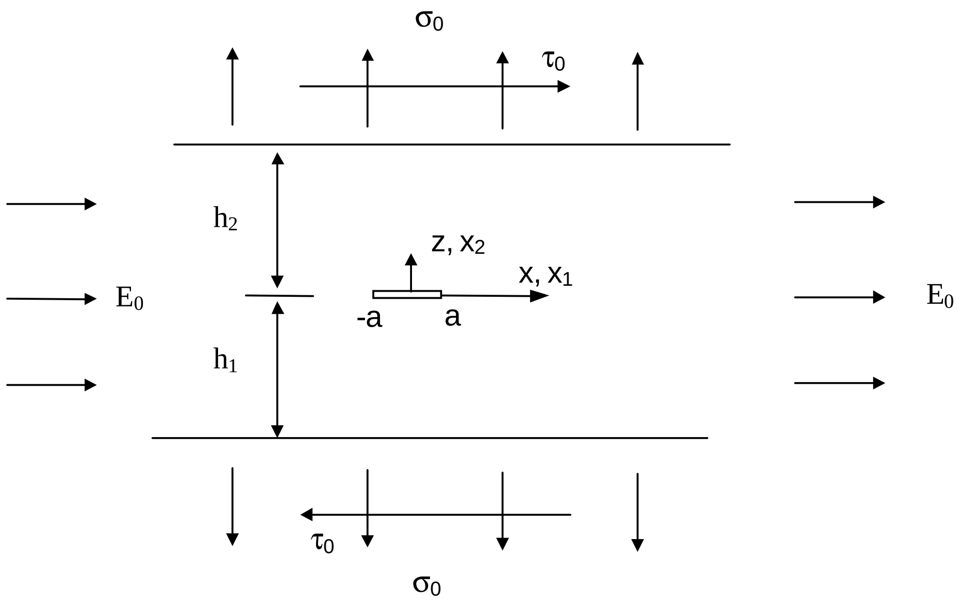

Fig.1. A cracked piezoelectric medium subjected to a remote electromechanical impulse.

In order to facilitate the analysis, constitutive Eq.(2.1) are re-written as

$$\begin{Bmatrix}\sigma_{22}\\ \sigma_{11}\\ \sigma_{12}\\ E_2\\ E_1\end{Bmatrix}=\begin{bmatrix}\bar{c}_{11} & \bar{c}_{13} & 0 & 0 & \bar{e}_{31}\\ \bar{c}_{13} & \bar{c}_{33} & 0 & 0 & \bar{e}_{33}\\ 0 & 0 & \bar{c}_{44} & \bar{e}_{15} & 0\\ 0 & 0 & \bar{e}_{15} & \bar{\in}_{11} & 0\\ \bar{e}_{31} & \bar{e}_{33} & 0 & 0 & \bar{\in}_{33}\end{bmatrix}\begin{Bmatrix}u_{2,2}\\ u_{1,1}\\ u_{1,2}+u_{2,1}\\ D_2\\ D_1\end{Bmatrix}. \tag{2.2}$$

Since the electric displacement D is divergence-free in the absence of a space charge, there exists a potential function $\Phi(x_1, x_2)$ such that

$$D_1=\Phi_{,2}, \qquad D_2=-\Phi_{,1}. \tag{2.3}$$

Further, it follows from $E_i=-\phi_{,i}$ that

$$E_{1,2} - E_{2,1} = 0 . \tag{2.4}$$

The mechanical equilibrium equations $\sigma_{11,1} + \sigma_{12,2} = \rho u_{1,tt}$ and $\sigma_{12,1} + \sigma_{22,2} = \rho u_{2,tt}$, and the electric field Eq.(2.4) can be written in terms of u_1, u_2 and Φ by inserting Eqs (2.3) and (2.2) into them, giving

$$\left.\begin{array}{l} \bar{c}_{33}u_{1,11} + \bar{c}_{44}u_{1,22} + (\bar{c}_{13} + \bar{c}_{44})u_{2,12} + (\bar{e}_{33} - \bar{e}_{15})\Phi_{,12} = \rho u_{1,tt} \\ (\bar{c}_{13} + \bar{c}_{44})u_{1,12} + \bar{c}_{44}u_{2,11} + \bar{c}_{11}u_{2,22} - \bar{e}_{15}\Phi_{,11} + \bar{e}_{31}\Phi_{,22} = \rho u_{2,tt} \\ (\bar{e}_{33} - \bar{e}_{15})u_{1,12} + \bar{e}_{31}u_{2,22} - \bar{e}_{15}u_{2,11} + \bar{\in}_{33}\Phi_{,22} + \bar{\in}_{11}\Phi_{,11} = 0 \end{array}\right\} \tag{2.5}$$

where ρ is the mass density.

In order to make the problem simplified, the displacements, electric potential, stresses, and electric field are assumed to be zero at the initial time. Let the piezoelectric medium be loaded suddenly by a shear stress $\sigma_{12} = \tau_0$ and a normal stress $\sigma_{22} = \sigma_0$ on the top and bottom surfaces, and an electric field $E_1 = E_0$ at infinity of the medium. For an electrically conducting crack, its surfaces are free from mechanical stresses and the electric field. Then the crack face boundary conditions can be stated as follows

$$\sigma_{12}(x,\ z=0,\ t) = \sigma_{22}(x,\ z=0,\ t) = E_1(x,\ z=0,\ t) = 0 , \qquad |x| < a , \tag{2.6}$$

At the cracked interface between the upper and lower media, the stress components σ_{12} and σ_{22} and the electric field component E_1 are continuous inside as well as outside the crack. The displacements u_1, u_2 and the potential function Φ are only continuous outside the crack.

3. Electro-elastic solutions

The Laplace transform technique will be used to solve the equilibrium equations. Hereafter, all of the field variables will represent the corresponding values in the Laplace transform domain. The solutions are obtained in terms of some unknown coefficients. These unknown coefficients are then determined by applying the boundary conditions (2.3) and by introducing two displacement discontinuity functions and an electric potential discontinuity function along the cracked plane ($z = 0$ plane). In the following analysis, we will use the following notations

$$\{b\} = \{b_1, b_2, b_3\}^T = \{\sigma_{12}, \sigma_{22}, E_1\}^T , \tag{3.1a}$$

$$\{b_0\} = \{b_{10}, b_{20}, b_{30}\}^T = \{\tau_0, \sigma_0, E_0\}^T , \tag{3.1b}$$

$$\{u\} = \{u_1, u_2, u_3\}^T = \{u_1, u_2, \Phi\}^T . \tag{3.1c}$$

We first consider a particular solution of Eq.(2.5)

$$u_i^* = a_i x + c_i z , \qquad i = 1, 2, 3 \tag{3.2}$$

where a_i and $c_i(i-1, 2, 3)$ are some real constants, which are determined from the boundary conditions (a shear stress $\sigma_{12} = \tau_0$ and a normal stress $\sigma_{22} = \sigma_0$ on the top and bottom surfaces, and an electric field $E_1 = E_0$ at infinity of the medium) in the Laplace transform domain. Here and in the sequel a variable with a superscript * represents its Laplace transform.

Next we consider a homogeneous solution of Eq.(2.5). We apply the Laplace transform to a time variable *t* and express the solution in terms of an unknown vector $\{F\} = (F_1, F_2, F_3, F_4, F_5, F_6)^T$ as follows

$$u_i^* = \frac{1}{2\pi}\int_{-\infty}^{\infty}\sum_{j=1}^{6} A_{ij}F_j \exp\left(|s|\lambda_j z\right)\exp\left(-\mathrm{i}sx\right)ds\ , \qquad i = 1, 2, 3 \tag{3.3}$$

where $\mathrm{i} = \sqrt{-1}$; A_{ij} are eigenvectors, and λ_j are eigenvalues of the following characteristic equation

$$\begin{bmatrix} \bar{c}_{33} - \bar{c}_{44}\lambda_m^2 + \dfrac{\rho p^2}{s^2} & \mathrm{i}\,sgn(s)(\bar{c}_{13} + \bar{c}_{44})\lambda_m & \mathrm{i}\,sgn(s)((\bar{e}_{33} - \bar{e}_{15}))\lambda_m \\ \mathrm{i}\,sgn(s)(\bar{c}_{13} + \bar{c}_{44})\lambda_m & \bar{c}_{44} - \bar{c}_{11}\lambda_m^2 + \dfrac{\rho p^2}{s^2} & -\bar{e}_{15} - \bar{e}_{31}\lambda_m^2 \\ \mathrm{i}\,sgn(s)(\bar{e}_{33} - \bar{e}_{15})\lambda_m & -\bar{e}_{15} - \bar{e}_{31}\lambda_m^2 & \bar{\epsilon}_{11} - \bar{\epsilon}_{33}\lambda_m^2 \end{bmatrix}\begin{Bmatrix} A_{1m} \\ A_{2m} \\ A_{3m} \end{Bmatrix} = 0 \tag{3.4}$$

and where "*p*" is the Laplace transform parameter. If the problem is a steady one, *p* is zero. Equation (3.4) is an eigenvalue problem. A nontrivial set (A_{1j}, A_{2j}, A_{3j}) exists if λ_j is a root of the determinant in Eq.(3.4). It is clear that there are six sets of roots for λ_j and the corresponding eigenvectors for (A_{1j}, A_{2j}, A_{3j}). For each part of the media above and below the crack, there is a homogeneous solution of the form Eq.(3.3).

The complete solutions for the displacements and electric potential are the sum of Eqs (3.2) and (3.3). Substituting them into the constitutive Eq.(2.2) in the Laplace transform domain, the following expressions for stresses and electric displacement can be obtained

$$b_i^* = \frac{1}{2\pi}\int_{-\infty}^{\infty} sB_{ij}F_j \exp\left(-\mathrm{i}sx\right)ds + b_{i0}^*\ , \qquad i = 1, 2, 3 \tag{3.5}$$

where summation over the index $j(j = 1, ..., 6)$ is assumed, $[B(s, z)]$ is a $3\acute{}6$ matrix which may be expressed in an analytical form by applying Eqs (3.2) and (3.3) to the constitutive relation (2.2).

Considering the surface boundary conditions of the medium and making use of the Fourier inversion to Eq.(3.5), the unknown coefficient $\{F\}$ can be expressed in terms of the Fourier transforms of the stresses σ_{12} and σ_{22} and the electric field E_1 on the cracked plane

$$\{F^+\} = \frac{1}{s}\left[G^+(s)\right]\left[\left(\tilde{\sigma}_{12}^*, \tilde{\sigma}_{22}^*, \tilde{E}_1^*\right)^T - \left(\tilde{\tau}_0^*, \tilde{\sigma}_0^*, \tilde{E}_0^*\right)^T\right], \tag{3.6a}$$

$$\{F^-\} = \frac{1}{s}\left[G^-(s)\right]\left[\left(\tilde{\sigma}_{12}^*, \tilde{\sigma}_{22}^*, \tilde{E}_1^*\right)^T - \left(\tilde{\tau}_0^*, \tilde{\sigma}_0^*, \tilde{E}_0^*\right)^T\right] \tag{3.6b}$$

where the signs "+" and "-" denote, respectively, the quantities in the media above and below the crack, G^+ and G^- are $6 ´ 3$ matrices (see Appendix A). Here and in the following, a variable with an over bar "~" represents its Fourier transform.

Substituting of Eqs (3.6a, b) into Eq.(3.3) leads to the solution of displacements u_1 and u_2 and the potential function Φ, in terms of the Fourier transforms of the stresses σ_{12} and σ_{22} and the electric field E_1 on the cracked plane. With Eq.(3.6a, b), u_1, u_2 and Φ given in Eq.(3.3) satisfy the continuity conditions for the stresses and electric field on the cracked plane.

4. Singular integral equations

We introduce the following discontinuity functions along the cracked interface

$$g_i(x) = \frac{\partial u_i^*\left(x, 0^+\right)}{\partial x} - \frac{\partial u_i^*\left(x, 0^-\right)}{\partial x}, \qquad i = 1, 2, 3. \tag{4.1}$$

The continuity conditions for the displacement and electric potential on the cracked plane require that

$$g_i(x) = 0 \quad \text{for} \quad |x| \geq a \quad \text{and} \quad \int_{-a}^{a} g_i(x)dx = 0, \qquad i = 1, 2, 3. \tag{4.2}$$

By substituting Eqs (3.5) into the Eq. (3.3), one obtains the displacements u_1, u_2 and the potential function Φ. They are then substituted into Eq.(4.1). This gives

$$g_i(r) = \frac{1}{2\pi i}\int_{-\infty}^{\infty}\left[C_{ij}(s)\right]\left(\tilde{b}_j^* - \tilde{b}_{j0}^*\right)\exp(-\mathrm{i}sr)ds \tag{4.3}$$

where C_{ij} are the elements of a 3 by 3 matrix C defined by

$$[C] = [A]\left(\left[G^+(s)\right] - \left[G^-(s)\right]\right) \tag{4.4}$$

and where $[A] = \left[A_{ij}\right]$. Equation (4.3) can be inversed to give

$$b_i^* = \int_{-a}^{a}\left[R_{ij}(x,\ r)\right]g_j(r)dr + b_{i0}^*, \qquad i = 1, 2, 3 \tag{4.5}$$

where summation over the index j is assumed, the kernel $[R(x, r)]$ is

$$[R(x,r)] = \frac{i}{2\pi}\int_{-\infty}^{\infty}[C(s)]^{-1}\exp[\mathrm{i}s(r-x)]ds\ . \tag{4.6}$$

In deriving the above equation, we have used conditions (4.2). By now g_i are the only unknowns in the problem, which may be determined from the boundary conditions on the crack faces. Note that for large values of s, $[C(s)]^{-1}$ becomes $-sgn(s)[\Lambda]$, where $[\Lambda]$ is a 3 by 3 constant matrix, which depends only on material properties. It follows from Eq.(4.6) that

$$[R(x,r)] = \frac{1}{\pi}\frac{[\Lambda]}{r-x} + [\Xi(x,r)], \tag{4.7}$$

$$[\Xi(x,r)] = \frac{1}{2\pi i}\int_{-\infty}^{\infty}\left([-C(s)]^{-1} - [\Lambda]\right)\exp[is(r-x)]ds. \tag{4.8}$$

Equation (4.5) gives σ_{12}, σ_{22} and E_1 outside as well as inside the crack. For the later, the boundary conditions (2.6) give

$$\frac{\Lambda_{ij}}{\pi}\int_{-a}^{a}\frac{1}{r-x}g_j(r)dr + \int_{-a}^{a}\Xi_{ij}(x,r)g_j(r)dr = -b_{i0}^*, \qquad i = 1, 2, 3. \tag{4.9}$$

The above singular integral equation contains a Cauchy-type kernel. Its solutions has the following form

$$g_i(a\bar{r}) = \frac{1}{\sqrt{1-\bar{r}^2}}\sum_{m=1}^{\infty}T_m(\bar{r})C_i(m), \qquad i = 1, 2, 3 \tag{4.10}$$

where $\bar{r} = r/a$, $\bar{x} = x/a$, T_m is the Chebyshev polynomial of the first kind, $C_i(m)$ are constants to be determined. Substitution of Eq.(4.10) into Eq.(4.9) leads to

$$\Lambda_{ij}\sum_{m=1}^{\infty}U_{m-1}(\bar{x})C_j(m) + a\int_{-1}^{1}\frac{\Xi_{ij}(x,r)}{\sqrt{1-\bar{r}^2}}\sum_{m=1}^{\infty}T_m(\bar{r})C_j(m)d\bar{r} = -b_{i0}^* \tag{4.11}$$

where summation over the index j is assumed, and U_{m-1} is the Chebyshev polynomial of the second kind. After evaluating g_i from Eq.(4.11), the stress and electric field intensity factors in the Laplace transform domain, i.e., in-plane shear (Mode II), in-plane normal traction (Mode I), and in-plane electric field (Mode E), $\{K\} = \{K_{II}, K_I, K_E^T\}$ can be calculated from

$$\{K\} = \left(\sqrt{2\pi[(-a)-x]}\right)_{x\to(-a)^-}(\sigma_{12}(x_1,0), \quad \sigma_{22}(x_1,0), \quad E_1(x_1,0))^T, \tag{4.12}$$

for the left-hand side crack-tip. The result is

$$\{K\} = [\Lambda]\sqrt{\pi a}\sum_{m=1}^{\infty}(-1)^m[C_1(m), \quad C_2(m), \quad C_3(m)]^T \quad . \tag{4.13a}$$

Similarly, for the right-hand side crack-tip, the field intensity factors are

$$\{K\} = -[\Lambda]\sqrt{\pi a}\sum_{m=1}^{\infty}[C_1(m), \quad C_2(m), \quad C_3(m)]^T \quad . \tag{4.13b}$$

5. Collinear cracks

In formulating the problem, no conditions of symmetry with respect to $x = 0$ were assumed regarding the crack geometry and the external loads. Thus, the integral Eq.(4.9) derived in Section 4 is valid basically for any number of collinear cracks defined by $z=0$, $b_i<x<c_j$, $(j=1,...,n)$ along the *x*-axis with the additional single-value condition of the form (3.5) for each crack, namely

$$g_{ij}(x)=0 \quad \text{for} \quad |x|\notin(b_j,c_j) \quad \text{and} \quad \int_{b_j}^{c_j} g_{ij}(x)dx=0, \qquad (i=1,2,3;\ j=1,...,n). \tag{5.1}$$

The only change in the integral equation would be in replacing the integral $(-a,a)$ by the sum of the integrals $L_i=(b_i,c_i)$ $(i=1,...,n)$ corresponding to the collinear cracks.

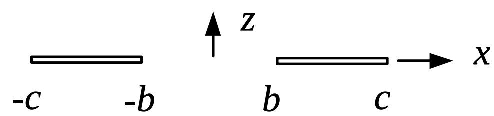

Fig.2. Two collinear cracks.

As an example, we consider the case of two symmetrically located and symmetrically loaded collinear cracks (Fig.2). That is, we assume that $b_1=b$, $c_1=c$, $b_2=-c$, $c_2=-b$, $\tau_{yz}(x,0)=0$. In this case, using the symmetry conditions, Eq.(4.9) may be expressed as

$$\frac{\Lambda_{ij}}{\pi}\int_b^c\left[\frac{1}{r-x}+\frac{1}{r+x}\right]g_j(r)dr+\int_b^c K_{ij}(x,r)g_j(r)dr=-b_{i0}^*, \qquad (i=1,2,3) \tag{5.2}$$

where summation over the index *j* is assumed, and

$$K(x,t)=\Xi(x,t)-\Xi(x,-t). \tag{5.3}$$

The integral Eq.(5.2) is again solved under the following single-valuedness condition

$$\int_b^c g_i(x)dx=0, \qquad (i=1,2,3). \tag{5.4}$$

By normalizing the length parameters according to

$$x=\bar{x}(c-b)/2+(c+b)/2, \qquad r=\bar{r}(c-b)/2+(c+b)/2, \tag{5.5}$$

the integral Eq.(5.2) can be reduced to the following standard form

$$\frac{\Lambda_{ij}}{\pi}\int_{-1}^{1}\left[\frac{1}{\bar{r}-\bar{x}}+\frac{1}{\bar{r}+\bar{x}+2\dfrac{c+b}{c-b}}\right]g_j(r)d\bar{r}+\frac{c-b}{2}\int_{-1}^{1}K_{ij}g_j(r)d\bar{r}=-b_{i0}^*, \qquad (i=1,2,3). \tag{5.6}$$

The solution of Eq.(5.6) has the same form as Eq.(4.10). By substituting g_i from Eq.(4.10) into Eq.(5.6) and by using the well-known orthogonality condition

$$\frac{1}{\pi}\int_{-1}^{1}\frac{T_m(\bar{r})}{(\bar{r}-\bar{x})\sqrt{1-\bar{r}^2}}d\bar{r}=\begin{cases} U_{n-1}(\bar{x}), & m\geq 1, \quad |\bar{x}|<1 \\ -\frac{sgn(\bar{x})}{\sqrt{\bar{x}^2-1}}\left[\bar{x}-sgn(\bar{x})\sqrt{\bar{x}^2-1}\right]^m, & m\geq 0, \quad |\bar{x}|>1 \\ 0, & m-0, \quad |\bar{x}|<1 \end{cases} \tag{5.7}$$

we find

$$\begin{aligned}&\Lambda_{ij}\sum_{m=1}^{\infty}U_{m-1}(\bar{r})C_j(m)+\Lambda_{ij}\sum_{m=1}^{\infty}C_j(m)\frac{1}{\sqrt{x_1{}^2-1}}\left[\sqrt{x_1{}^2-1}-x_1\right]^m+\\&+\frac{c-b}{2}\sum_{m=1}^{\infty}\int_{-1}^{1}K_{ij}(x,r)\frac{T_m(\bar{r})}{\sqrt{1-\bar{r}^2}}C_j(m)d\bar{r}=-b_{i0}^*\end{aligned} \tag{5.8}$$

where

$$x_1=\bar{x}+\frac{2(c+b)}{c-b}. \tag{5.9}$$

The simplest method for solving the functional Eq.(5.9) is truncating the series and using an appropriate collocation in *x*. In this problem, the stress and electric displacement intensity factors can also be obtained from Eqs (4.13a) and (4.13b).

The preceding sections establish the solution in the Lapalce transform domain. The corresponding values of the elastic and electric fields in the time domain are given by the Laplace inversion. This is achieved by adopting the numerical technique outlined in Miller and Guy (1996) which has been widely used in fracture dynamics (Sih and Chen, 1981). Further, the steady solution can be easily obtained by setting the Laplace transform parameter "*p*" to zero.

Once the stress and electric field intensity factors in the time domain are obtained, the energy release rate at the crack tips can be obtained from the virtual crack closure integral. This gives

$$G=\frac{1}{4}\{K_{II},K_I,K_E\}\left[\Lambda^{-1}\right]\{K_{II},K_I,K_E\}^T. \tag{5.10}$$

6. The electrode solution

If the region $x_1\in(-a,a)$, $x_2=0$ (or $x_1\in(-b,c)$, $x_2=0$) is not a crack but a rigid electrode, the above solution procedure in sections 4 and 5 is still valid. The only difference is that the auxiliary function $g_1(x_1)$ and $g_2(x_1)$ should be zero because the rigid electrode cannot be mechanically opened. In this situation, the singular integral Eq.(4.9) takes the following form

$$\frac{\Lambda_{33}}{\pi}\int_{-a}^{a}\frac{1}{r-x}g_3(r)\,dr+\int_{-a}^{a}\Xi_{33}(x,r)g_3(r)\,dr=-b_{30}^* \tag{6.1}$$

where Λ_{33} and Ξ_{33} are elements of the matrices $[\Lambda]$ and $[\Xi]$, respectively.

The singular integral for two collinear electrodes is obtained from Eq.(5.6) by setting $g_1(x_1) = g_2(x_1) = 0$. The result is

$$\frac{\Lambda_{22}}{\pi}\int_{-1}^{1}\left[\frac{1}{\bar{r}-\bar{x}}+\frac{1}{\bar{r}+\bar{x}+2\dfrac{c+b}{c-b}}\right]g_3(r)\,d\bar{r}+\frac{c-b}{2}\int_{-1}^{1}K_{33}\,g_3(r)d\bar{r}=-b_{30}^{*}. \tag{6.2}$$

The solution method of the singular integral Eqs (6.1) and (6.2) is same as that of Eqs (4.9) and (6.2). Note that in this problem the solution depends only on the applied electric field load. Since g_1 and g_2 are zero, then $C_1(m)$ and $C_2(m)$ in Eq.(4.10) are zero. It follows from Eqs (4.13a, b) that the electric field intensity factors are

$$K_{\mathrm{E}}(b)=\sqrt{\pi a}\Lambda_{33}\sum_{n=1}^{\infty}(-1)^n C_3(n), \qquad K_{\mathrm{E}}(c)=-\sqrt{\pi a}\Lambda_{33}\sum_{n=1}^{\infty}C_3(n). \tag{6.3}$$

The stress intensity factor is

$$K_{II}=\frac{\Lambda_{13}}{\Lambda_{33}}K_E, \qquad K_I=\frac{\Lambda_{23}}{\Lambda_{33}}K_E. \tag{6.4}$$

In this problem, the energy release rate is still given by Eq. (5.10). It can be shown from Eqs (6.4) and (5.10) that

$$G=\frac{1}{4}\frac{1}{\Lambda_{33}}K_E^2. \tag{6.5}$$

7. Numerical example

A cracked PZT-4 piezoelectric strip is taken as a numerical example. The thickness of the strip is *h*. A crack of length *2a* is located at the center of the strip. Sudden electromechanical loads σ_0 and E_0 are applied simultaneously to the medium. No shear load is considered here because τ_0 alone does not cause a coupled electro-mechanical field near the crack tip. The density of the medium is denoted by ρ. The electro-mechanical properties of the medium are as follows (Fulton and Gao, 1997; Gao *et al*., 1997)

$$c_{11}=13.9\left(10^{10}\,N/m^2\right), \qquad c_{13}=7.43\left(10^{10}\,N/m^2\right),$$

$$c_{33}=11.3\left(10^{10}\,N/m^2\right), \qquad c_{44}=2.56\left(10^{10}\,N/m^2\right),$$

$$e_{31}=-6.98C/m^2, \quad e_{33}=13.8C/m^2 \quad e_{15}=13.4C/m^2,$$

$$\in_{11}=60.0\left(10^{-10}\,C/Vm\right), \qquad \in_{33}=54.7\left(10^{-10}\,C/Vm\right)$$

where *m*, *N*, *C*, and *V* are length in meter, force in Newton, charge in Coulomb and electric potential in volt, respectively. The elements of the matrix $[\Lambda]$ are obtained as $\Lambda_{11} = 2.164\times10^{10}$, $\Lambda_{12} = 0$, $\Lambda_{13} = 0$, $\Lambda_{22} = 2.838\times10^{10}$, $\Lambda_{23} = 0.05338\times10^{10}$, $\Lambda_{33} = 0.005375\times10^{10}$.

Due to the symmetry of the loading conditions, the mode II stress intensity factor is zero. The applied stress and electric field intensity factors are defined as $\sigma_0\sqrt{\pi a}$ and $E_0\sqrt{\pi a}$, respectively. In what follows, we focus our attention on the steady crack tip field for a strip of finite thickness, and the dynamics crack tip field for an infinite medium.

7.1. Steady crack tip field

Table 1 gives the steady values of the stress and electric field intensity factors for different crack length to strip thickness ratios. It is well known that for an infinity medium, the stress and electric field intensity factors are not coupled. This is, an applied stress does not cause the electric field intensity factor, and an applied electric field does not produce the stress intensity factor. However, the results in table 1 show that for a medium of finite size, the stress and electric fields ahead of the crack tip are coupled. However, an applied stress load can produce a large electric field intensity factor, but the stress intensity factor caused by an applied electric field load is negligible. Take $H/a = 2$ as an example, an applied stress intensity of $K_I = 1MPa(m)^{-3/2}$ can produce a crack tip electric field intensity factor of $K_E = 15.18\,KV/(m)^{-3/2}$, and an applied electric field intensity of $K_E = 40\,KV/(m)^{-3/2}$ can only produce a crack tip stress intensity factor of $-0.01273\,MPa(m)^{-3/2}$.

Table 1. Steady stress and electric field intensity factors and energy release rate for different strip thickness.

H/a		2	4	8	16	30	Infinity
$E_0 = 0$	$K_I/(\sigma_0\sqrt{\pi a})$	2.010	1.322	1.087	1.023	1.0065	1
	$K_E/(\sigma_0\sqrt{\pi a})$	0.01518	0.004691	0.001273	3.256´10^{-4}	9.314´10^{-5}	0
$\sigma_0 = 0$	$K_I/(E_0\sqrt{\pi a})$	-0.3183	-0.1659	-0.05737	-0.01581	-0.004606	0
	$K_E/(E_0\sqrt{\pi a})$	1.176	1.056	1.015	1.004	1.001	1

The energy release rate can be obtained from Eq.(5.10), using the stress and intensity factors shown in Tab.1. The resulting different strip thickness are as follows

$$H/a = 2: \qquad G = \pi a\left(0.3849\sigma_0^2 - 3.169\sigma_0 E_0 + 79.88E_0^2\right)\times10^{-10},$$

$$H/a = 3: \qquad G = \pi a\left(0.2307\sigma_0^2 - 2.690\sigma_0 E_0 + 68.80E_0^2\right)\times10^{-10},$$

$$H/a = 4: \qquad G = \pi a\left(0.1771\sigma_0^2 - 2.482\sigma_0 E_0 + 64.15E_0^2\right)\times10^{-10},$$

$$H/a = 8: \qquad G = \pi a\left(0.1255\sigma_0^2 - 2.243\sigma_0 E_0 + 59.06E_0^2\right)\times10^{-10},$$

$$H/a = 16: \qquad G = \pi a\left(0.1126\sigma_0^2 - 2.175\sigma_0 E_0 + 57.67E_0^2\right)\times10^{-10},$$

$$H/a = 30: \qquad G = \pi a\left(0.1095\sigma_0^2 - 2.158\sigma_0 E_0 + 57.33E_0^2\right)\times10^{-10},$$

$$H/a = \infty: \qquad G = \pi a\left(0.1083\sigma_0^2 - 2.151\sigma_0 E_0 + 57.19E_0^2\right)\times10^{-10}.$$

The influence of medium thickness on the values of G is very significant.

7.2. Dynamic crack tip field

Numerical results for a dynamics loaded infinity medium are obtained. The stress and electric field intensity factors at crack tips are expressed as follows

$$K_I(t)=[f_{\sigma\sigma}(t)\sigma_0+f_{\sigma E}(t)E_0]\sqrt{\pi a}\,,\qquad K_E(t)=[f_{E\sigma}(t)\sigma_0+f_{EE}(t)E_0]\sqrt{\pi a} \tag{7.1}$$

where the coefficients ***f*** depend on the geometry and material properties, but not on the applied loads σ_0 and E_0.

The time-varying values of the coefficients in Eq.(7.1) are plotted in Figs 3-6. Each of these curves increases with time from the initial zero value, displays a peak value and then decreases to the steady value as time becomes large enough. Unlike the steady stress intensity factor, which is independent of the applied electric field load, the transient stress intensity factors can increase or decrease with the electric field load, depending on time t. It can be shown that the dynamic applied electric field can produce significant stress intensity factors at the crack tips. Therefore, even for the infinity piezoelectric medium, the transient stress and electric fields are strongly coupled at the cracked tips.

Finally, Fig.7 displays the transient energy release rate G for the different electric field loads applied. For the electric field loads between $E_0/\sigma_0=-0.04$ and 0.02, G is a monotonously decreasing function of E_0. This fact suggests that a positive electric field will retard and a negative electric field will enhance crack growth, at all times.

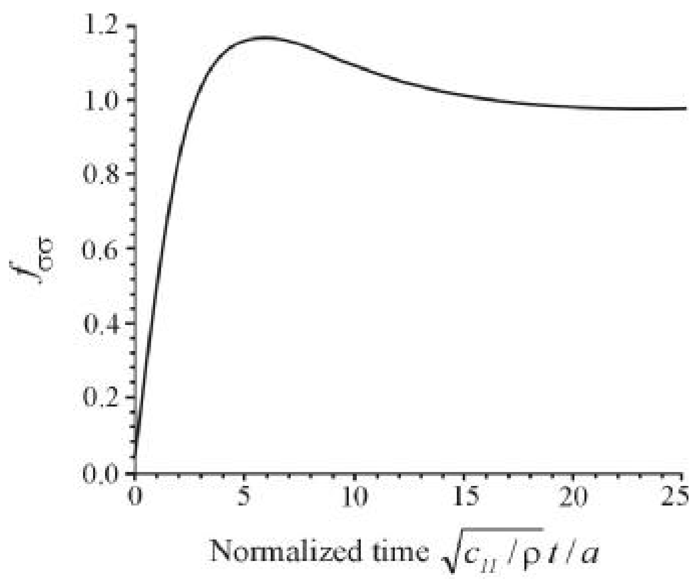

Fig.3. Normalized stress intensity factors caused by stress load.

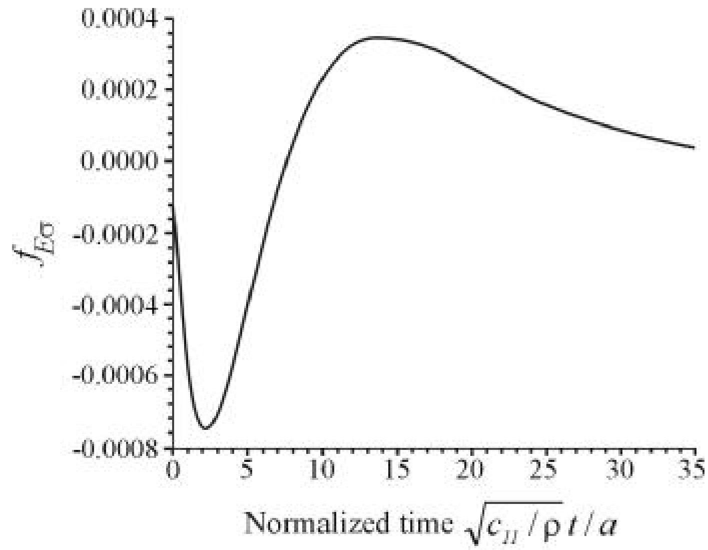

Fig.4. Normalized electric field intensity factors caused by stress load.

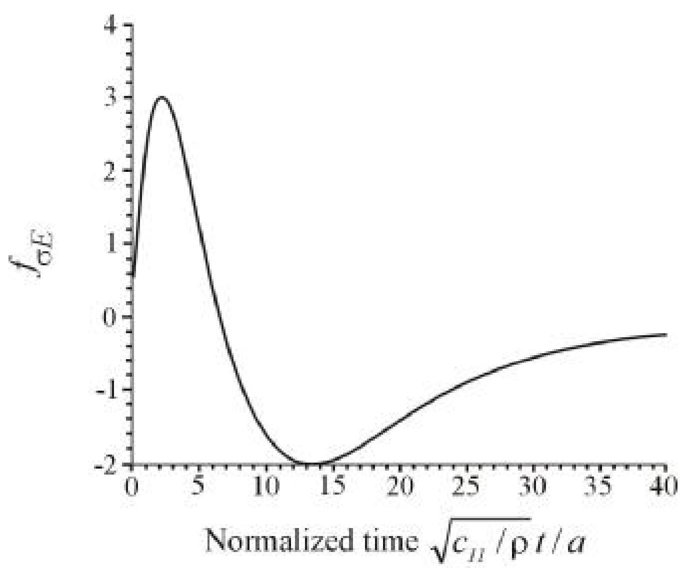

Fig.5. Normalized stress field intensity factors caused by electric field load.

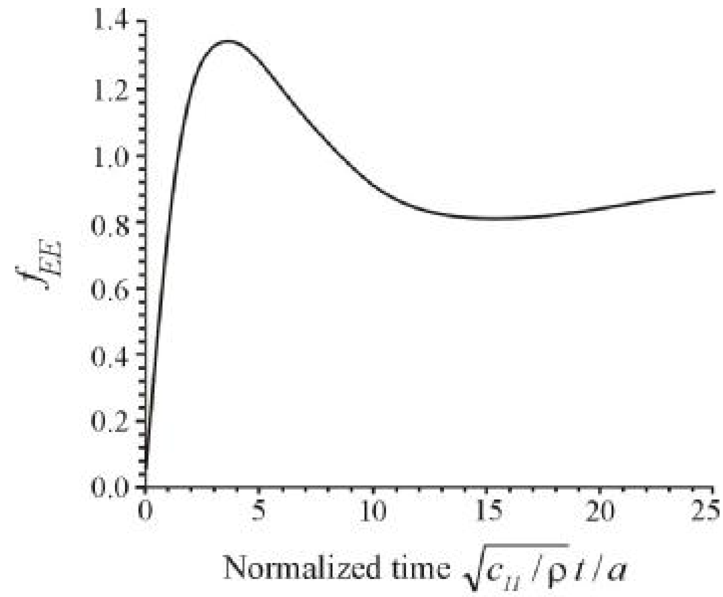

Fig.6. Normalized electric field intensity factors caused by electric field load.

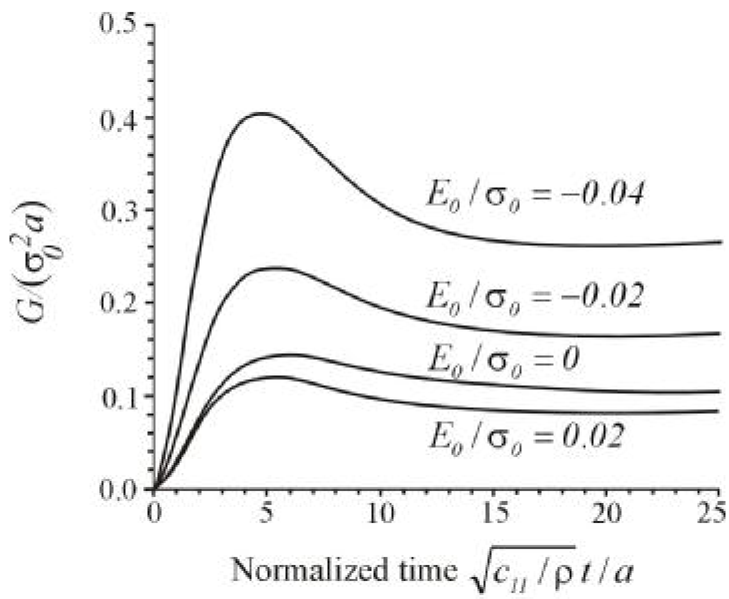

Fig.7. Normalized energy release rate for different electric field loads.

7.3. Electrode solution

The electrode tip fields response only to the applied electric field load. The stress and electric field intensity factors have the following forms

$$K_{\mathrm{I}} = \frac{\Lambda_{23}}{\Lambda_{33}} K_E = f_{\sigma E}(t) E_0 \sqrt{\pi a}\,, \qquad K_E(t) = f_{EE}(t) E_0 \sqrt{\pi a} \tag{7.2}$$

where the values of f_{EE} obtained for an infinite electrode are similar to those for an infinite crack, which are given in Fig.6. The coefficient $f_{\sigma E}$ plotted in Fig.8 suggests that for an applied electric field load, the stress intensity factor K_I at an electrode tip is considerably larger than that at a crack tip (see Fig.5). Further, the values of K_I at the electrode tip increase with the electric field loads for any values of time *t*.

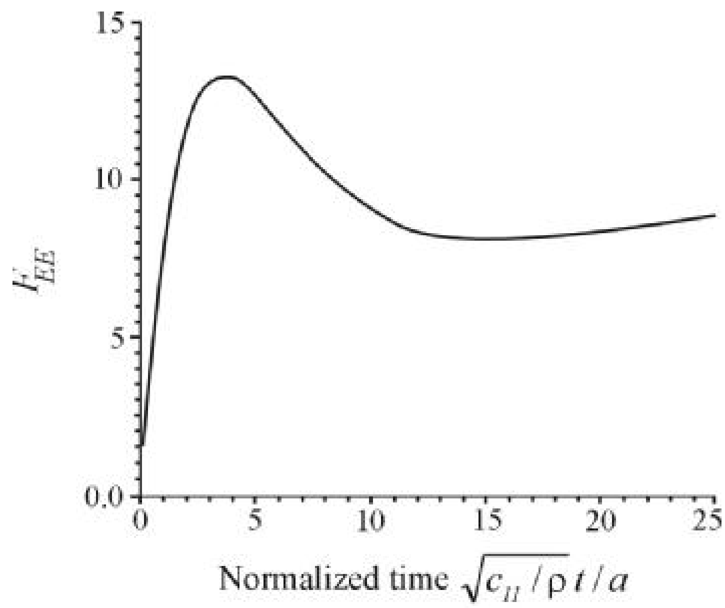

Fig.8. Normalized electric field intensity factors for an infinity electrode.

8. Conclusions

This paper considers conducting crack or electrodes in piezoelectric media under an in-plane electro-mechanical impact. The stress and electric field intensity factors and the energy release rate are obtained. The quantitative results obtained in this paper suggest that for a piezoelectric medium of finite size or under a transient loading condition, the electric load can produce stress intensity factors at the crack tips, and vice versa, i.e., the mechanical load can also produce electric field intensity factors. At different time, crack tip stress intensity factors can increase or decrease with the electric field loads, depending on the direction of the applied electric fields. The electrode tip stress intensity factors increase with the applied electric field loads, at all times.

Acknowledgments

We are grateful to the Australian Research Council (ARC) for the financial support of this work through the Discovery Projects (#DP0346037 and #DP0665856). BLW and YWM are, respectively, Australian Research Fellow and Australian Federation Fellow supported by the ARC and tenable at the CAMT of the University of Sydney.

Nomenclature

c – elastic constant
D – electric displacement

e – piezoelectric constant
G – energy release rate
K_E – electric field intensity factor
K_{II}, K_I – stress intensity factors
t – time
u – displacement
ρ – density
σ – stress
ϕ – electric potential
$\in$ – dielectric permittivities

References

Fulton C.C. and Gao H-J. (1997): *Electrical nonlinearity in fracture of piezoelectric ceramics*. – App. Mech. Rev., vol.50, No.11, part 2, pp.1-8.

Gao H-J., Zhang T.Y. and Tong P. (1997) *Local and global energy release rates for an electrically yielded crack in a piezoelectric ceramic*. – J. Mech. Phys. Solids, vol.45, No.4, pp.491-510.

He T.H. (2002): *Steady propagate crack in a transverse isotropic piezoelectric material considering the permittivity of the medium in the crack gap*. – Int. J. Fracture, vol.118, No.3, pp.239-249.

Jin B., Soh A.K. and Zhong Z. (2003): *Propagation of an anti-plane moving crack in a functionally graded piezoelectric strip*. – Arch. Appl. Mech., vol.73, No.3-4, pp.252-260.

Kwon S.M. and Lee K.Y. (2003): *Steady state crack propagation in a piezoelectric layer bonded between two orthotropic layers*. – Mech. Mater., vol.35, No.11, pp.1077-1088.

Li X.F. and Tang G.J. (2003): *Transient response of a piezoelectric ceramic strip with an eccentric crack under electromechanical impacts*. – Int. J. Solids Struct., vol.40, No.13-14, pp.3571-3588.

Miller M.K. and Guy W.T. (1996): *Numerical inversion of the Laplace transform by use of Jacobi polynomials*. – SIAM J. Numer. Anal., vol.3, pp.624-635.

Nishioka T., Shen S.P. and Yu J.H. (2003): *Dynamic J integral, separated dynamic J integral and component separation method for dynamic interfacial cracks in piezoelectric biomaterials*. – Int. J. Fracture, vol.122, No.3-4,: pp.101-130.

Ricci V., Shukla A., Chalivendra V.B and Lee K.H. (2003): *Subsonic interfacial fracture using strain gages in isotropic-orthotropic biomaterial*. – Theor. Appl. Fract. Mec., vol.39, No.2, pp.143-161.

Sih G.C. and Chen E.P. (1981): *Cracks in Composite Materials*. – In: Mechanics of Fracture (ed, Sih, G.C.) Martinus Nijhoff, The Hague.

Sosa H. and Khutoryansky N. (1999): *Transient dynamics response of piezoelectric bodies subjected to internal electric impulses*. – Int. J. Solids and Structures, vol.36, pp.5467-5484.

Sosa H. and Khutoryansky N. (2001): *Further analysis of the transient dynamics response of piezoelectric bodies subjected to electric impulses*. – Int. J. Solids and Structures, vol.38, pp.2101-2114.

Ueda S. (2003): *Transient dynamic response of a coated piezoelectric strip with a vertical crack*. – Eur. J. Mech. A-Solid, vol.22, No.6, pp.925-942.

Appendix A

Consider the medium above the crack. On the cracked plane, $z = 0$, $b_i^* = \left(\tilde{\sigma}_{12}^*, \tilde{\sigma}_{22}^*, \tilde{E}_1^*\right)^T$. It follows from Eq.(3.5) that

$$\frac{1}{2\pi}\int_{-\infty}^{\infty} sB_{ij}(s,\ z=0)F_j \exp(-\mathrm{i}sx)ds = \left(\tilde{\sigma}_{12}^*,\ \tilde{\sigma}_{22}^*,\ \tilde{E}_1^*\right)^T - \left(\tilde{\tau}_0^*,\ \tilde{\sigma}_0^*,\ \tilde{E}_0^*\right)^T . \tag{A1}$$

On the top surface, the vector b_i^* is zero. Hence from Eq.(3.5) we have

$$\frac{1}{2\pi}\int_{-\infty}^{\infty} sB_{ij}(s,\ z=h_2)F_j \exp(-\mathrm{i}sx)ds = 0, \qquad i=1,2,3. \tag{A2}$$

From Eqs (A1) and (A2), F_j can be obtained as Eq.(3.6a) in which the six by three matrix G^+ is

$$\left[G^+(s)\right] = \begin{bmatrix} B_{ij}(s,0) \\ B_{ij}(s,h_2) \end{bmatrix}^{-1} \begin{bmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \\ 0 & 0 & 0 \\ 0 & 0 & 0 \\ 0 & 0 & 0 \end{bmatrix}. \tag{A3}$$

Similarly, the six by three matrix G^- is obtained as

$$\left[G^-(s)\right] = \begin{bmatrix} B_{ij}(s,0) \\ B_{ij}(s,-h_1) \end{bmatrix}^{-1} \begin{bmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \\ 0 & 0 & 0 \\ 0 & 0 & 0 \\ 0 & 0 & 0 \end{bmatrix}. \tag{A4}$$

Received: March 30, 2004
Revised: December 14, 2004