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THERMAL BOUNDARY LAYER ON AN EXPONENTIALLY STRETCHING CONTINOUS SURFACE IN THE PRESENCE OF MAGNETIC FIELD EFFECT

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The thermal boundary layer on an exponentially stretching continuous surface with an exponential temperature distribution in the presence of the magnetic field effect is investigated numerically. The local similarity solution is applied to the governing equations. Comparisons with previously published work are made and the results are found to be in excellent agreement. Numerical results for temperature distribution and the local Nusselt number have been presented for different values of the governing parameters. In particular, it has been found that the magnetic field decreases the temperature difference at the wall of the stretching surface, while the Nusselt number decreases with it.

Key words: stretching surface, magnetic field, boundary layer, similarity solutions.

1. Introduction

The problem of heat transfer in the boundary layer induced by a continuous stretching surface with a given temperature distribution in a quiescent conducting fluid is important in several manufacturing processes in industry. Examples of such processes are the extrusion of plastic sheets, glass-fiber and paper production, metal spinning and the cooling of a metallic plate in a cooling bath. After Sakiadis (1961), many authors have studied the problem of flow induced by a surface moving with constant velocity (Tsou *et al.*, 1967; Griffin and Throne, 1967; Grubka and Bobba, 1985). Recently, Ali (1994) studied the thermal boundary layer of a continuous stretching surface. The similarity solutions of flow and thermal boundary layer on an exponentially stretching surface are studied by Magyari and Keller (1999). These solutions involve an exponential dependence of the temperature distribution in the direction parallel to that of the stretching.

The study of magnetohydrodynamics of a conducting fluid finds applications in a variety of astrophysical and geophysical problems. The effects of the magnetic field on the natural convection heat

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transfer have been discussed by Romig (1964), Elbashbeshy (1998), considered heat transfer over a stretching surface with a variable surface heat flux. The convective heat transfer in an electrically conducting fluid at a stretching surface has been studied by Vajravelu and Hadjinicolaou (1997). Other studies dealing with hydromagnetic flows can be found in Grandet *et al.* (1992), Takhar and Ram (1994), and Duwairi and Damseh (2003).

When the combined effects of the magnetic and fluid forces are incorporated into the governing equations of an exponentially stretching surface with an exponential temperature distribution, the analytical solutions as well as the similarity solutions become intractable. The aim of the present paper is to introduce a local similarity solution of an exponentially stretching surface with an exponential dependence of the temperature distribution in the presence of the magnetic field effect. Numerical solutions are obtained to study the characteristics of the thermal boundary layer in terms of different governing parameters.

2. Governing equations

Consider a two-dimensional flow of an electrically conducting and incompressible viscous fluid near an impermeable plane wall stretching with velocity U_w and a given temperature distribution T_w, the *x*-axis is taken along the wall in the upward direction and the *y*-axis perpendicular to it into the fluid. A uniform magnetic field B_0 is assumed to be applied in the *y*-direction. It is assumed that the induced magnetic field of the flow is negligible in comparison with the applied one which corresponds to a very small magnetic Reynolds number (Pai, 1962). Under boundary layer approximation, the continuity, momentum, and energy equations can be written as

$$\frac{\partial u}{\partial x} + \frac{\partial v}{\partial y} = 0\,, \tag{2.1}$$

$$u\frac{\partial u}{\partial x} + v\frac{\partial u}{\partial y} = \nu\frac{\partial^2 u}{\partial y^2} - \frac{\sigma B_o^2}{\rho}u \tag{2.2}$$

$$u\frac{\partial T}{\partial x} + v\frac{\partial T}{\partial y} = \alpha\frac{\partial^2 u}{\partial y^2} + \frac{\sigma\ B_o^2}{\rho c_p}u^2\,. \tag{2.3}$$

This system of Eqs (2.1)-(2.3) is subjected to the following boundary conditions

$$\left.\begin{array}{lll} u = U_w,\ v = 0,\ T = T_w & \text{at} & y = 0 \\ u = 0, \quad T = T_\infty, & \text{at} & y = \infty \end{array}\right\} \tag{2.4}$$

u and v are the x and y components of the velocity field, respectively. υ denotes the kinematic viscosity, α is the thermal diffusivity, σ is the electrical conductivity and B_o is the magnetic field flux density. The stretching velocity U_w and the exponential temperature distribution are defined as

$$U_w(x) = U_o e^{x/L}\,, \tag{2.5}$$

$$T_w(x) = T_\infty + (T_o - T_\infty)e^{ax/2L}\,. \tag{2.6}$$

If the magnetic field influence is absent, the folowing parameters can be applied to Eqs (2.1)-(2.4) to obtain similarity solutions (Magyari and Keller, 1999)

$$u(x,y)=U_o e^{x/L} f'(\eta), \tag{2.7}$$

$$v(x,y)=-\frac{v}{L}\sqrt{\mathrm{Re}/2}\, e^{x/2L}\left[f(\eta)+\eta f'(\eta)\right], \tag{2.8}$$

$$T(x,y)=T_\infty+(T_o-T_\infty)e^{ax/2L}\,\theta(\eta), \tag{2.9}$$

$$\eta=\sqrt{\mathrm{Re}}\,\frac{y}{L}e^{x/2L}. \tag{2.10}$$

In the present problem, considering the magnetic field effects, the similarity parameters (Eqs (2.7)-(2.10) can be used to transform Eqs (2.1)-(2.4) to η local similarity transformations. Thus, the governing equations using the dimensionless functions $f(\eta)$ and $\theta(\eta)$ become

$$f''' + ff'' - 2f'^2 - \frac{2\mathrm{Ha}^2}{\mathrm{Re}}e^{-X} f' = 0, \tag{2.11}$$

$$\frac{1}{\mathrm{Pr}}\theta'' + f\theta' - af'\theta + \frac{2\mathrm{Ha}^2\mathrm{Ec}}{\mathrm{Re}}e^{X(2-a)/2} f'^2 = 0. \tag{2.12}$$

The corresponding boundary conditions transform to

$$\left.\begin{aligned} &f(0)=0, \qquad f'(0)=1, \qquad f'(\infty)=0, \\ &\theta(0)=1, \qquad \theta(\infty)=0 \end{aligned}\right\} \tag{2.13}$$

In the above equations the prime denotes derivatives with respect to η. $\mathrm{Ha}=\left(\sigma B_o L^2/\rho\upsilon\right)^{1/2}$, $\mathrm{Ec}=U_0^2/c_p(T_0-T_\infty)$, $\mathrm{Re}=U_o L/\upsilon$, $\mathrm{Pr}=\upsilon/\alpha$ denotes the Hartman number, Eckert number, Reynolds number, Prandtl number respectively. In the above system of local similarity equations, the effect of the magnetic field is included as a ratio of the Hartman number to the Reynolds number. In addition to f' and θ, the local Nusselt number and local skin-friction coefficient are important physical parameters for this problem. These can be defined as

$$\tau_{wx}=\rho\upsilon\frac{\partial u}{\partial y}(x,0)=\frac{\rho\upsilon U_o}{L}\sqrt{\mathrm{Re}/2}\, e^{X/2} f''(0), \tag{2.14}$$

$$\mathrm{Nu}_x=\frac{q_w(x)x}{k(T_w(x)-T_\infty)}=-\sqrt{\mathrm{Re}_x}\left(\frac{X}{2}\right)^{1/2}\theta'(0). \tag{2.15}$$

3. Solution methodology

The set of nonlinear differential Eqs (2.11) and (2.12) with the appropriate boundary conditions in Eq. (2.13) is a boundary value problem which has unknown analytical solutions. There are different well-established numerical techniques to solve such equations. For this purpose, a FORTRAN computer program is developed to solve the above set of differential equations. The finite difference method with a non uniform grid is applied, the BVPFD subroutine in the IMSL-library which is assigned to solve the boundary value problems is requested by the main FORTRAN program for each trial. The solution domain is identified and the value of η_∞ at different X location is assigned.

A grid independence study was carried out to examine the effect of the step size η, X on the solution. According to the optimization study computations were carried out using a uniform grid in the X-direction with $\Delta X = 0.001$. Instead of using uniform grids in the η-direction, a non-uniform grid is incorporated with the first step size $\Delta\eta = 0.005$ and the variable grid parameter is chosen to be *1.02*. Under relaxation is required to secure convergence of the iteration procedure. The range for the under relaxation factor is taken as *0.1-0.6* for the velocity and temperature fields. The convergence criterion in iteration is stated as

$$\left|\frac{\psi_b - \psi_a}{\psi_b}\right| \le \varepsilon \tag{3.1}$$

where ψ_b and ψ_a denote one of the main variables f, θ, and the subscripts b and a denote the values corresponding to the new iteration and old iteration. The value for the tolerance ε is taken as 10^{-6}.

In order to assess the accuracy of our methods as described earlier, we have compared the temperature profiles for $a = -1.5$, $\Pr = 0.5$, 3, 8 and the value of $\mathrm{Ha}^2/\mathrm{Re}$ is assigned 0 according to the results of Magyari and Keller (1999) employing the shooting method in Fig.1. It can be seen that the results are in good agreement. Furthermore, the wall temperature gradient $\theta'(0)$ for different values of a and Pr and in the absence of the magnetic field effect is compared with the results of Magyari and Keller (1999) in Tab.1. The small difference may, however, be attributed to the different methods used.

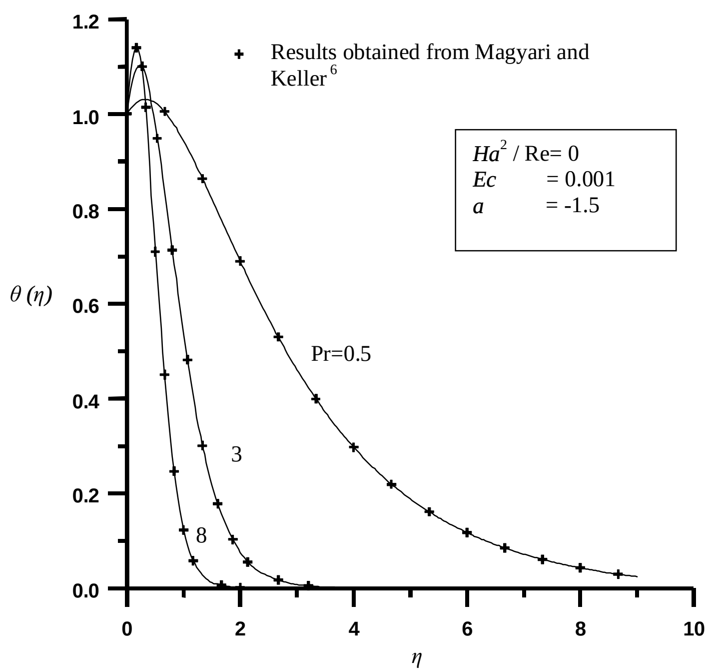

Fig.1. Comparison between dimensionles temperature profiles obtained in this study and those of Magyari and Keller (1999). ($a=-1.5$, $\mathrm{Ha}^2/\mathrm{Re}=0$ and different Prandtl numbers).

Table 1. Comparison between wall-temperature gradient calculated by the present method $(^{**})$ and that of Magyari and Keller (1999) $(^{*})$, in the absence of magnetic field effect $(Ha=0)$.

Pr / *a*		*0.5*	*1*	*3*	*5*	*8*	*10*
-1.5	*	*0.204049*	*0.377413*	*0.923857*	*1.353240*	*1.888500*	*2.200000*
	**	*0.191914*	*0.361516*	*0.903084*	*1.341428*	*1.828580*	*2.136932*
-0.5	*	*-0.175815*	*-0.299876*	*-0.634113*	*-0.870431*	*-1.150321*	*-1.308613*
	**	*-0.181869*	*-0.326974*	*-0.672150*	*-0.841562*	*-1.083914*	*-1.250740*
0	*	*-0.330493*	*-0.549643*	*-1.122188*	*-1.521243*	*-1.991847*	*-2.257429*
	**	*-0.310061*	*-0.531044*	*-1.085222*	*-1.475581*	*-1.926328*	*-2.188474*
1	*	*-0.594338*	*-0.954782*	*-1.869075*	*-2.500135*	*-3.242129*	*-3.660379*
	**	*-0.577705*	*-0.919033*	*-1.810391*	*-2.288641*	*-3.005874*	*-3.186202*
3	*	*-1.008405*	*-1.560294*	*-2.938535*	*-3.886555*	*-5.000465*	*-5.628198*
	**	*-0.976654*	*-1.465689*	*-2.890073*	*-3.780721*	*-4.862453*	*-5.585759*

It is of interest to note that the transformed energy Eq.(2.12) in the absence of the applied magnetic flux $(B_o=0)$ and for $a=0$, is reduced to a simple flat plate heat transfer problem for constant wall temperature. Such an equation, which is coupled with the simple Blasius equation, is solved by the method presented here and compared with the shooting method used by White (1991). The comparisons are not presented here but they show a good agreement which emphasizes the accuracy of the selected method.

4. Results and discussion

In the present study, numerical calculations are performed in terms of the temperature distribution and the local Nusselt number for different values of aforementioned physical parameters. Referring to the governing Eqs. (2.11) and (2.12) the temperature distribution through the thermal boundary layer and the local Nusselt number depend on Pr, a, $\mathrm{Ha}^2/\mathrm{Re}$, X location. The value of the Eckert number Ec is given a positive value equal to 0.001 for all the predicted results. Positive values of Ec mean that the reference temperatures T_o must be greater than the free stream temperature $T_\infty (T_o > T_\infty)$ which is the case considered in the present problem. If $T_o - T_\infty > 0$ (the considered case) then, according to Eq.(2.6), the wall temperature is greater than the free stream temperature and heat is transferred from the wall to the fluid.

The dimensionless temperature field and the local Nusselt number depend extensively on a, this effect is shown in Figs 2 and 3 for X location $=0.5$, $\mathrm{Ha}^2/\mathrm{Re}=5$, $\mathrm{Pr}=1$. The simplest case is the constant wall temperature $(a=0)$. It is clear from Fig.2 that the heat transfer process is reversed at some value of a, i.e., the flow of heat is directed from the wall to the ambient environment (the lower curves) and then reversed from the ambient environment to the wall (the upper curves). Due this, there is an adiabatic case where the heat transfer process is stopped. The value of a in the adiabatic case can be examined analytically.

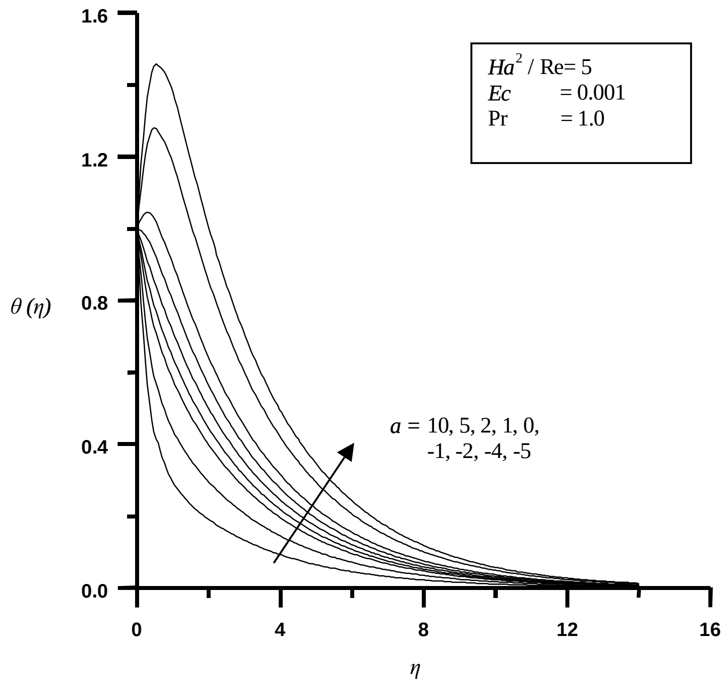

Fig.2. Dimensionles temperature profiles at dimensionless $X = 0.5$, $\Pr = 1$ and $\mathrm{Ha}^2/\mathrm{Re} = 5$ for different values of *a*.

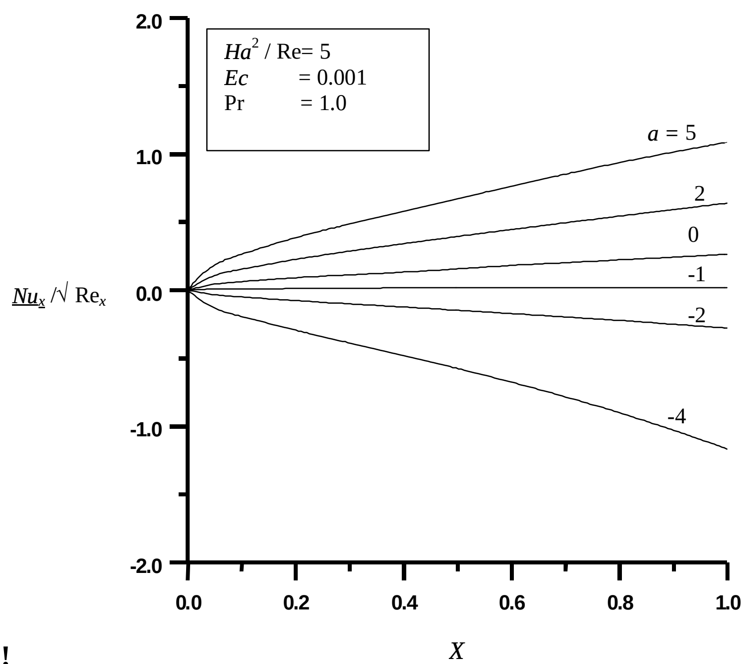

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Fig.3. Local Nusselt number and the dimensionless X location at $\Pr = 1$, $\mathrm{Ha}^2/\mathrm{Re} = 5$ for different values of *a*.

Using Eqs (2.13), (2.12) can be integrated once from $\eta = 0$ to ∞, this gives

$$\theta'(0) = \Pr \frac{2\mathrm{Ha}^2\mathrm{Ec}}{\mathrm{Re}} \int_0^\infty e^{X(2-a)/2} f'^2(\eta) d\eta - \Pr(a+1) \int_0^\infty \theta(\eta) f'(\eta) d\eta \,. \tag{4.1}$$

Equation (4.1) shows that the slope at the stretching surface represented by $\theta'(0)$ depends on four parameters, Pr, a, X and the dimensionless group $\mathrm{Ha}^2\mathrm{Ec}/\mathrm{Re}$. The case of no heat transfer between the stretched surface and the ambient fluid corresponds to $\theta'(0)=0$, this case cannot be determined analytically. Negative values of $\theta'(0)$ correspond to heat transfer from the stretched surface to the ambient fluid, the reversed case occurred at positive values of $\theta'(0)$. The problem of no magnetic field effect $(\mathrm{Ha}=0)$ is discussed by Magyari and Keller (1999), for such a problem the adiabatic case is satisfied at $a=-1$ for any value of Pr, then the negative slope occurred for $a>-1$ whereas, for $a<-1$ the reversed heat transfer process occurred. Anyhow, from Fig.2 and at selected values of Pr, X, $\mathrm{Ha}^2\mathrm{Ec}/\mathrm{Re}$ it can be clearly observed that heat transfer is increased by decreasing the value of a below the adiabatic value, the presence of the peak indicates that the maximum value of temperature occurs in the body of the fluid close to the surface and not at the surface. It is interesting to note that this increase in the temperature accompanied by a greater increase in the temperature peak value which increases the temperature difference between the stretched wall and the adjacent fluid is the reason for triggering the heat transfer process from the ambient fluid to the surface. On the other hand, increasing the value of a above the adiabatic value leads, also, to increasing the heat transferred from the wall to the ambient fluid. This behavior can be seen in Fig.3. The increase in the local Nusselt number in both discussed cases (a above and below the adiabatic value) is clearly noticed. Also, Fig.3 shows the increase in the local Nusselt number by moving away from the leading edge of the stretched surface, this result can be clearly obtained from Eq.(2.15). Figure 4 depicts the influence of the X location on the temperature distribution through the thermal boundary layer for $a=-3$ and -6 (below the adiabatic value). Obviously, the peak is getting larger by moving away from the leading edge. As explained above, the heat transfer is enhanced; this result is as concluded from Fig.3.

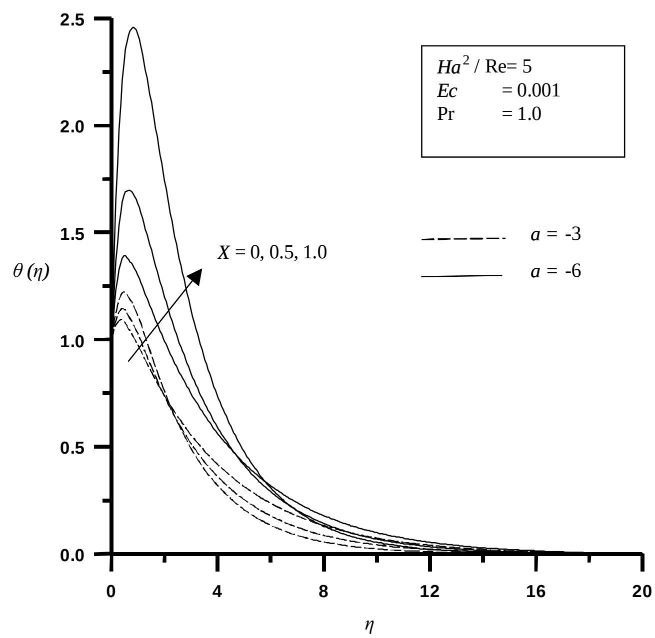

Fig.4. Dimensionles temperature profiles for $a = -3$ and -6, $\Pr = 1$, $\mathrm{Ha}^2/\mathrm{Re} = 5$ at different dimensionless x location.

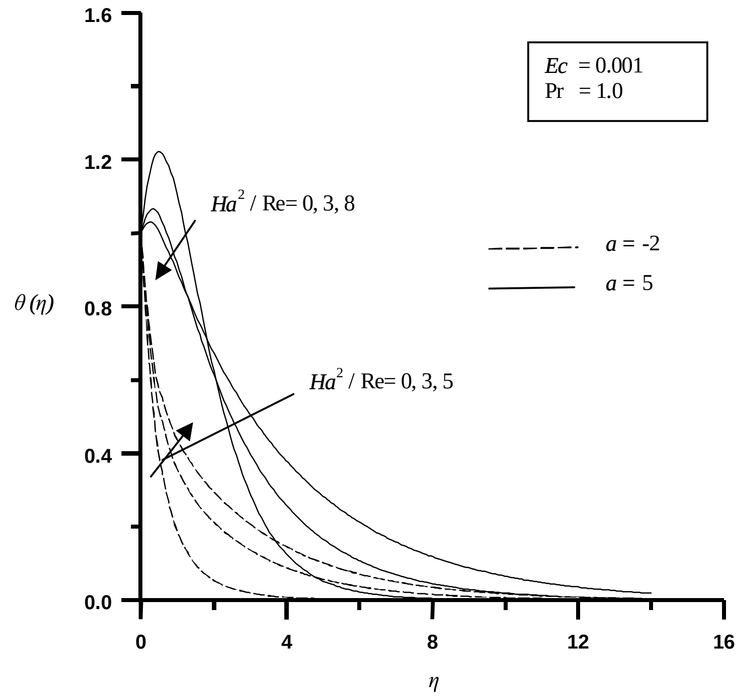

Fig.5. Influence of magnetic field on dimensionless temperature profiles at $X = 0.5$, $\Pr = 1$, $a = -2$ and 5.

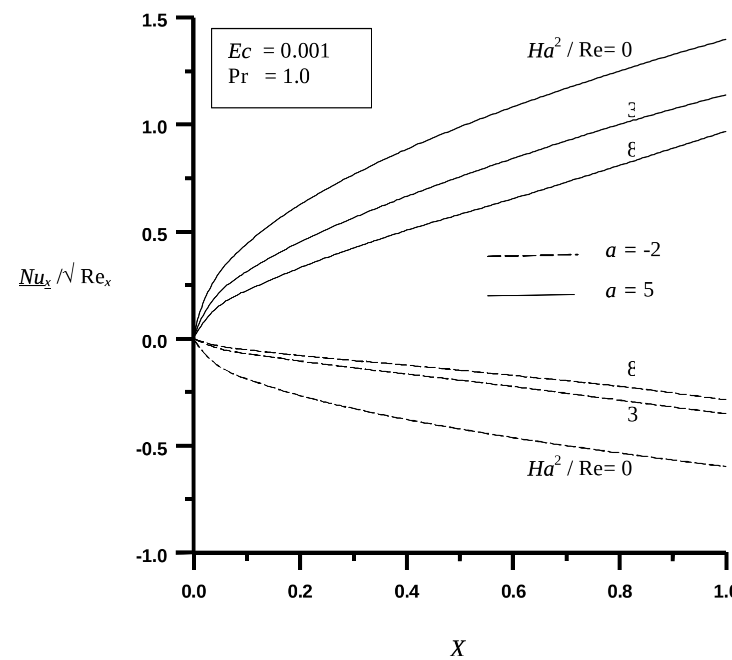

Fig.6. Effect of magnetic field on Local Nusselt number distribution at $\mathrm{Pr}=1$, $a=-2$ and 5.

Figure 5 presents the temperature distribution in the fluid for some selected values of $\mathrm{Ha}^2/\mathrm{Re}$ at $a=-2$ and 5. For $a=-2$, it can be seen that an increase in the strength of the magnetic field (represented by the Hartmann number) leads to an increase in the thermal boundary thickness, so the temperature inside the thermal boundary layer increases due to excess heating. This will decrease the temperature difference at the wall of the stretched surface. The same effect is noticed for the second $(a=5)$, i.e., the decrease in the temperature near the wall which can be predicted by a decrease in the peak value mentioned before. For both cases, the flow of heat is decreased due to an increase in the strength of the magnetic field. Figure 6 illustrates the coclusions drawn from Fig.5. On the other hand, it is clear from this figure that moving away from the stretched surface (in the *X*-direction) results in a greater heat flow, this is assessed as an increase in the Nusselt number.

5. Conclusions

Numerical solutions for the thermal boundary layer of an exponentially continuous stretching surface with exponential temperature variations at the wall in the presence of the magnetic field have been examined. The following conclusions has been made.

1. The dimensionless temperature field and the local Nusselt number depend on Pr, *a*, *X* and the dimensionless group $\mathrm{Ha}^2\mathrm{Ec}/\mathrm{Re}$. There are three cases defining the flow of heat which depends upon the above parameters. The case of no heat transfer (adiabatic case), this case can be determined analytically in the absence of the magnetic field effect which corresponds to $a=-1$. The second case corresponds to *a* value above the adiabatic one; in this case the heat transfer is directed from the wall to the ambient fluid (direct flow). Lastly, the flow of heat is directed from the ambient fluid towards the wall (reversed flow); this case corresponds to the value of *a* below the adiabatic value.
2. The thickness of the thermal boundary layer increases with decreasing *a* for constant Pr, *X* and the dimensionless group $\mathrm{Ha}^2\mathrm{Ec}/\mathrm{Re}$. Increasing *a* (above the adiabatic value) as well as decreasing it (below the adiabatic value) enhances the heat transfer process.
3. The thickness of the thermal boundary layer increases with increasing the strength of the magnetic field in both heat transfer cases (direct and reversed flow), for both cases the local Nusselt number decreases with increasing the magnetic field strength.
4. The effect of *X* location along the plate decreases the thickness of the thermal boundary layer and increases the heat transfer rate.

Nomenclature

a – constant
B_o – magnetic field flux density
c_p – specific heat of the fluid at constant pressure
Ec – Eckert number
Ha – Hartmann number
L – length of the plate
Nu – Nusselt number
Pr – Prandtl number
Re – Reynolds number
T – fluid temperature
T_∞ – ambient temperature
T_o – reference temperature

T_w – wall temperature
u – fluid axial velocity
U_o – reference velocity
U_w – velocity of the vertical surface
v – fluid transverse velocity
X – dimensionless coordinate along the plate (x/L)
x, y – coordinates along and normal to the plate, respectively
υ – kinematic viscosity
τ_w – coefficient of skin friction
α – thermal diffusivity
η – non-dimensional transformed variable
σ – fluid electrical conductivity
ρ – fluid density
θ – dimensionless temperature

Subscripts

x – local
w – conditions on the wall
o – reference
∞ – ambient conditions

Superscripts

$'$ – differentiation with respect to η

References

Ali M.E. (1994): *Heat transfer characteristics of a contious stretching surface.* – Warme Stoffuberttragung, vol.29, pp.227-234.

Duwairi H.M. and Damseh R.A. (2004): *Magnetohydrodynamic natural convection heat transfer from radiate vertical porous surfaces*. – Heat Mass Transfer, vol.40, pp.787-792.

Elbashbeshy E.M.A. (1998): *Heat transfer over a stretching surface with variable surface heat flux*. – J. Phys. D: Appl. Phy., vol.31, pp.1951-1955.

Grandet J. P., AlboussiereT. and Moreau R. (1992): *Buoyancy driven convection in a rectangular enclosure with a transverse magnetic field*. – Int. J. Heat Mass Transfer, vol.35, pp.741-749.

Griffin J.F. and Throne J.L. (1967): *On thermal boundary layer groth on continous moving belts*. – Am. Inst. Chem. Eng. J., vol.13, pp.1210-1211.

Grubka L.J. and Bobba K.M. (1985): *Heat transfer characteristics of a continous stretching surface with variable temperature.* – ASME J. Heat Transfer, vol.107, pp.248-250.

Magyari E. and Keller B. (1999*): Heat and mass transfer in the boundary layers on an exponentially stretching continuous surface*. – J. Phys. D: Appl. Phy., vol.32, pp.577-585.

Pai S.I. (1962): *Magnetogasdynamics and Plasma Dynamic*. – Springer Verlag, Berlin.

Romig M. (1964): *The influence of electric and magnetic field on heat transfer to electrically conducting fluids.* – Adv. Heat Transfer, vol.1, pp.268-352.

Sakiadis B.C. (1961): *Boundary-layer behavior on continous solid surfaces: II. The boundary layer on a continous flat surface*. – AIChE J., vol.7, pp.221-225.

Takhar H.S. and Ram P. C. (1994): *Magnetohydrodynamic free convection flow of water at 4 ^{o}C through a porous medium.* – Int. Comm. Heat Mass Transfer, vol.21, pp.371-376.

Tsou K., Sparrow E.M. and Goldstein R.J. (1967): *Flow and heat transfer in the boundary layer on a continous moving surface*. - Int. J. Heat Mass Transfer, vol.10, pp.219-235.

White Frank M. (1991): *Viscous Fluid Flow.* – McGraw-Hill, Inc. New York.

Vajravelu K. and Hadjinicolaou A. (1997): *Convective heat transfer in an electrically conducting fluid at a stretching surface with uniform free streem.* – Int. J. Eng. Sci., vol.35, pp.1237-1244.

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