Int. J. of Applied Mechanics and Engineering, 2006, vol.11, No.2, pp.321-335

BOUNDARY LAYER VISCOELASTIC FLUID FLOW OVER AN EXPONENTIALLY STRETCHING SHEET

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In the present paper, a viscoelastic boundary layer fluid flow over an exponentially stretching continuous sheet has been examined. The flow is assumed to be generated solely by the application of two equal and opposite forces along the *x*-axis such that stretching of the boundary surface is of exponential order in *x*. Approximate analytical similarity solutions (zero and first order) of the highly non-linear boundary layer equation are obtained for the dimensionless stream function and velocity distribution function after transforming the boundary layer equation into Riccati type and solving that sequentially. The first-order solution is derived in the form of confluent hypergeometric Whittaker functions. The solutions are verified at the boundary sheet. These solutions (zero and first order) involve an exponential dependence of the similarity variable, the stretching velocity and the stream function on the axial coordinate. The accuracy of the analytical solutions is also verified by the numerical solutions obtained by employing the Runge-Kutta fourth order method with shooting. The effects of various physical parameters on the velocity profile and skin-friction coefficient are also discussed in this paper.

Key words: viscoelastic fluid, boundary layer flow, exponential stretching, skin-friction and Whittaker solution.

1. Introduction

Beginning with the pioneering work of Sakiadis (1961) on stretching sheet there has been a great deal of works on various aspects of momentum and heat transfer characteristics in a viscoelastic boundary layer fluid flow over a stretching plastic boundary (Rajagopal *et al*., 1984 and 1987; Dandapat and Gupta, 1989; Rollins and Vajravelu, 1991; Andersson, 1992; Lawrence and Rao, 1992; Char, 1994 and Rao, 1996). Some of the typical applications of such study are the polymer sheet extrusion from a dye, glass fiber and paper production, drawing of plastic films, etc. Extensive literature is also available including those cited above on the two-dimensional viscoelastic boundary layer fluid flow over a stretching surface where the velocity of the stretching surface is assumed linearly proportional to the distance from a fixed origin. Mcleod and Rajagopal (1987) discussed uniqueness of the solution of such fluid flow in case of a viscous fluid. Troy *et al*. (1987) discussed uniqueness of the solution of the boundary layer equation arising in a viscoelastic fluid flow past a linearly stretching sheet. However, it is often argued that (Gupta and Gupta, 1977) realistically stretching of a sheet may not necessarily be linear. This situation has been beautifully dealt with by Kumaran and Ramanaiah (1996) in their work on boundary layer flow where, probably first time, a general quadratic stretching sheet has been assumed. They analysed their results in terms of stream function. However, their work was confined to the viscous fluid flow over a stretching sheet.

Recently Ali (1995) investigated thermal boundary layer by considering a power law stretching surface. A new dimension was added to this investigation by Elbashbeshy (2001) who examined the flow and heat transfer characteristics by considering an exponentially stretching continuous surface. The Elbashbeshy (2001) considered an exponential similarity variable and exponential stretching velocity distribution on the coordinate considered in the direction of stretching. However, the works of Ali (1995) and Elbashbeshy (2001) are confined to the study of a viscous fluid flow only.

In reality, most of the fluids considered in industrial applications are more non-Newtonian in nature, especially of viscoelastic type than viscous type. To take into account these we extend the work of Elbashbeshy (2001) to the viscoelastic fluid flow in the boundary layer region. Approximate analytical similarity solutions (zero and first order) are obtained for the stream function by transforming the highly non-linear differential equation into Riccati type and then solving this sequentially. The aim of the article is to analyse the accuracy of the zero-order and first-order solutions by comparing these with the numerical solution obtained by the Runge-Kutta fourth order method with shooting. The effects of various physical parameters like the viscoelastic parameter and Reynolds number on the skin friction coefficient are also analysed.

2. Formulation of the problem

2.1. Preliminaries

The constitutive equation of an incompressible second order fluid is given by

$$\boldsymbol{T} = -p\boldsymbol{I} + \mu \boldsymbol{A}_1 + \alpha_1 \boldsymbol{A}_2 + \alpha_2 \boldsymbol{A}_1^2 \tag{2.1}$$

where $\boldsymbol{T}$ is the stress tensor, p is the pressure, μ is the dynamic viscosity, α_1, α_2 are the normal stress moduli and kinematical tensors $\boldsymbol{A}_1$ and $\boldsymbol{A}_2$ are defined by

$$\begin{aligned} \boldsymbol{A}_1 &= (\text{grad}\,\boldsymbol{q}) + (\text{grad}\,\boldsymbol{q})^T, \\ \boldsymbol{A}_2 &= \frac{d\boldsymbol{A}_1}{dt} + \boldsymbol{A}_1 \cdot (\text{grad}\,\boldsymbol{q}) + (\text{grad}\,\boldsymbol{q})^T \cdot \boldsymbol{A}_1 . \end{aligned} \tag{2.2}$$

Equation (2.1) was derived by Coleman and Noll (1960) using the postulates of gradually fading memory. Using some experimental data verification Fosdick and Rajagopal (1979) gave the range of values of μ, α_1 and α_2 as

$$\mu \ge 0, \qquad \alpha_1 \le 0, \qquad \alpha_1 + \alpha_2 \ne 0 . \tag{2.3}$$

Making use of Eq.(2.1) Beard and Walters (1964) derived the steady state two-dimensional boundary layer equation for a viscoelastic fluid flow in the form

$$u\frac{\partial u}{\partial x} + v\frac{\partial u}{\partial y} = \gamma \frac{\partial^2 u}{\partial y^2} - k_0 \left\{ u\frac{\partial^3 u}{\partial x \partial y^2} + v\frac{\partial^3 u}{\partial y^3} - \frac{\partial u}{\partial y}\frac{\partial^2 u}{\partial x \partial y} + \frac{\partial u}{\partial x}\frac{\partial^2 u}{\partial y^2} \right\}. \tag{2.4}$$

This equation has been derived with the assumption that the normal stress is of the same order of magnitude as that of the shear stress, in addition to the usual boundary layer approximations.

2.2. Flow governing equations

The physical problem consists of a steady state two-dimensional boundary layer flow of an incompressible viscoelastic fluid of the type Walters liquid B over a stretching sheet (Fig.1). In formulating the problem we consider the following assumption.

(i) The boundary sheet is assumed to be moving axially with a velocity of exponential order in the axial direction and generating the boundary layer type of flow.

For the above physical consideration of the problem the governing boundary layer equations for momentum (Rajagopal *et al*., 1984 and Sonth *et al*., 2002) are

$$\frac{\partial u}{\partial x}+\frac{\partial v}{\partial y}=0\,, \tag{2.5}$$

$$u\frac{\partial u}{\partial x}+v\frac{\partial u}{\partial y}=\gamma\frac{\partial^2 u}{\partial y^2}-k_0\left\{u\frac{\partial^3 u}{\partial x\partial y^2}+v\frac{\partial^3 u}{\partial y^3}-\frac{\partial u}{\partial y}\frac{\partial^2 u}{\partial x\partial y}+\frac{\partial u}{\partial x}\frac{\partial^2 u}{\partial y^2}\right\} \tag{2.6}$$

where u and v are the velocity components in the x and y directions respectively, γ is the kinematics coefficient of viscosity, k_0 is the elastic parameter. Other quantities have their usual meanings (Abel ***et al.***, 2002 and Dandapat and Gupta, 1989). In deriving Eq.(2.6) it is assumed that

(i) The normal stress is of the same order of magnitude as that of the shear stress, in addition to the usual boundary layer approximations.

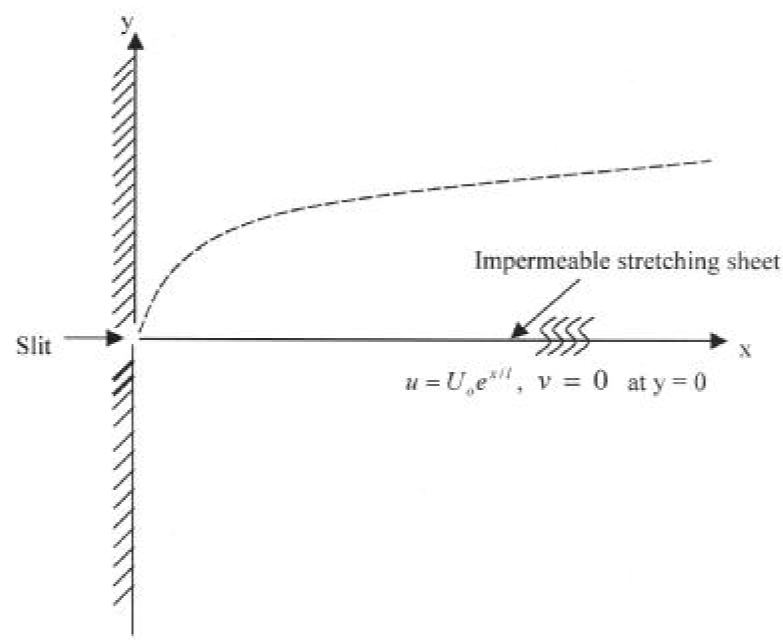

Fig.1. Boundary layer over an impermeable exponentially stretching sheet.

2.3. Boundary conditions on velocity

The boundary sheet is assumed to be stretched with a large force in such a way that stretching velocity along the axial direction x is of exponential order of the directional coordinate. Hence, we employ the following boundary conditions on velocity.

$$u=U_w(x)=U_0\exp\left(\frac{x}{l}\right) \qquad \text{at} \qquad y=0\,,$$

$$v=0 \qquad \text{at} \qquad y=0\,, \tag{2.7}$$

$$u=0 \qquad \text{as} \qquad y\to\infty$$

where U_0 is a constant and l is the reference length.

3. Solution of the momentum boundary layer equation

Equation (2.6) may be rewritten in terms of the stream function $\psi(x, y)$, which satisfies the equation of continuity (2.5), by writing

$$u = \frac{\partial \psi}{\partial y}, \qquad v = \frac{\partial \psi}{\partial x}. \tag{3.1}$$

Further, the stream function $\psi(x, y)$ may be non-dimensionalised by assuming

$$\psi(x, y) = \sqrt{2\gamma l U_0}\, f(\eta) \exp\left(\frac{x}{2l}\right), \tag{3.2}$$

$$\eta = y\sqrt{\frac{U_0}{2\gamma l}} \exp\left(\frac{x}{2l}\right). \tag{3.3}$$

Here f is the dimensionless stream function and η is the similarity variable. Making use of Eqs (3.1)-(3.2) in Eq.(2.6) we obtain a fourth order non-linear ordinary differential equation of the form

$$2 f_\eta^2 - f f_{\eta\eta} = f_{\eta\eta\eta} - k_1^* \left[3 f_\eta f_{\eta\eta\eta} - \frac{1}{2} f f_{\eta\eta\eta\eta} - \frac{3}{2} f_{\eta\eta}^2 \right]. \tag{3.4}$$

Here, $k_1^* = \dfrac{k_0 U_w}{\gamma}$ is the dimensionless viscoelastic parameter.

The boundary conditions on f are

$$\begin{aligned} &f = 0, \quad f_\eta = 1 \quad \text{at} \quad \eta = 0, \\ &f_\eta = 0 \quad \text{as} \quad \eta \to \infty. \end{aligned} \tag{3.5}$$

Integrating Eq.(3.4), we obtain

$$f_{\eta\eta} + f f_\eta = -S + \int_0^\eta \left[3 f_\eta^2 + k_1^* \left\{ 3 f_\eta f_{\eta\eta\eta} - \frac{1}{2} f f_{\eta\eta\eta\eta} - \frac{3}{2} f_{\eta\eta}^2 \right\} \right] d\eta \tag{3.6}$$

where $S = f_{\eta\eta}(0)$.

For $\eta \to \infty$, we get

$$S = -\int_0^\infty \left[3 f_\eta^2 + k_1^* \left\{ 3 f_\eta f_{\eta\eta\eta} - \frac{1}{2} f f_{\eta\eta\eta\eta} - \frac{3}{2} f_{\eta\eta}^2 \right\} \right] d\eta. \tag{3.7}$$

We integrate Eq.(3.6) once again and apply boundary conditions of Eq.(3.5). This yields

$$f_\eta + \frac{1}{2} f^2 = 1 - S\eta + \int_0^\eta \left[\int_0^{\eta_2} 3 f_{\eta_1}^2 + k_1^* \left(3 f_{\eta_1} f_{\eta_1\eta_1\eta_1} - \frac{1}{2} f f_{\eta_1\eta_1\eta_1\eta_1} - \frac{3}{2} f_{\eta_1\eta_1}^2 \right) d\eta_1 \right] d\eta_2 . \tag{3.8}$$

Now, the solution procedure of the Eq.(3.8) may be reduced to the sequential solutions of the Riccati-type equations

$$f_\eta^{(n)} + \frac{1}{2} f^{(n)2} = R \cdot H \cdot S\left(f_\eta^{(n-1)},\ f_{\eta\eta}^{(n-1)},\ f_{\eta\eta\eta}^{(n-1)},\ f_{\eta\eta\eta\eta}^{(n-1)} \right). \tag{3.9}$$

This iteration algorithm has to be solved by substituting suitable zero-order approximation $f_\eta^{(0)}(\eta)$ for $f_\eta(\eta)$ on the $R \cdot H \cdot S$ of Eq.(3.8).

We assume zero-order approximation of as $f_\eta^{(0)}(\eta)$ as

$$f_\eta^{(0)}(\eta) = \exp(-S_0\eta), \tag{3.10}$$

which satisfies the boundary conditions at infinity. Integrating the Eq.(3.10) and making use of boundary condition at $\eta = 0$ of the Eq.(3.5) we get

$$f^{(0)}(\eta) = \frac{1 - \exp(-S_0\eta)}{S_0} . \tag{3.11}$$

Substituting all the derivatives of zero-order approximation $f^{(0)}(\eta)$ into $R \cdot H \cdot S$ of Eq.(3.8) and assuming that first-order iteration $f^{(1)}(\eta)$ on the $L \cdot H \cdot S$ of Eq.(3.8) satisfying all the boundary conditions at $\eta = 0$ of Eq.(3.5) we obtain the value of S as

$$S_0 = \sqrt{\frac{3}{2\left(1 - k_1^*\right)}} \qquad \text{and} \qquad f_{\eta\eta}^{(0)}(0) = -S_0 . \tag{3.12}$$

Here the equation for first-order iteration $f_\eta^{(1)}(\eta)$ takes the form

$$f_\eta^{(1)}(\eta) + \frac{1}{2} f^{(1)2}(\eta) = 1 + \frac{\left(3 + k_1^* S_0^2\right)}{4S_0^2}\left(e^{-2S_0\eta} - 1\right) + \frac{k_1^*}{2}\left(e^{-S_0\eta} - 1\right). \tag{3.13}$$

Equation (3.13) is the non-linear Riccati equation and this can be solved for $f^{(1)}(\eta)$. In this solution process we introduce the following transformation.

$$f^{(1)}(\eta) = \frac{2U_\eta(\eta)}{U(\eta)} . \tag{3.14}$$

Hence, Eq.(3.13) is transformed to

$$U_{\eta\eta}(\eta)-\frac{1}{2}\left\{1+\frac{\left(3+k_1^* S_0^2\right)}{4S_0^2}\left(e^{-2S_0\eta}-1\right)+\frac{1}{2}k_1^*\left(e^{-2S_0\eta}-1\right)\right\}U(\eta)=0\,. \tag{3.15}$$

Equation (3.15) may be solved in terms of confluent hypergeometric Whittaker function in the following form.

$$U(\eta)=\frac{A}{\exp\left(-\frac{S_0}{2}\eta\right)}\left[M_{k,\frac{\mu}{2}}\left(2\sqrt{b_{11}}\exp(-S_0\eta)\right)+BW_{k,\frac{\mu}{2}}\left(2\sqrt{b_{11}}\exp(-S_0\eta)\right)\right] \tag{3.16}$$

where A and B are arbitrary constants. The functions $M_{k,\mu}(z)$ and $W_{k,\mu}(z)$ are the Whittaker's functions and they are the independent solutions of the Whittaker's equation

$$\frac{d^2q}{dz^2}+\left[-\frac{1}{4}+\frac{k}{2}+\frac{\frac{1}{4}-\mu^2}{z^2}\right]q=0 \tag{3.17}$$

where

$$W_{k,\mu}(z)=\frac{\Gamma(-2\mu)}{\Gamma\left(\frac{1}{2}-\mu-k\right)}M_{k,\mu}(z)+\frac{\Gamma(2\mu)}{\Gamma\left(\frac{1}{2}+\mu-k\right)}M_{k,-\mu}(z),$$

(3.18a)

$$M_{k,\mu}(z)=\exp\left(-\frac{z}{2}\right)z^{\frac{1}{2}+\mu}M\left(\frac{1}{2}+\mu-\kappa,1+2\mu,z\right).$$

Here, $M(a,b,z)$ is the Kummer's function (Abramowitz and Stegun, 1970) and it is defined by

$$M(a,b,z)=1+\sum_{n=1}^{\infty}\frac{(a)_n z^n}{(b)_n n!}\,,$$

$$(a)_n=a(a+1)(a+2)\ldots\ldots\ldots(a+n-1), \tag{3.18b}$$

$$(b)_n=b(b+1)(b+2)\ldots\ldots\ldots(b+n-1).$$

Substituting the solution of $U(\eta)$ of Eq.(3.16) in Eq.(3.14) and making use of the boundary condition $f^{(1)}(\eta)=0$ at $\eta=0$ we obtain the first-order approximate solution for $f(\eta)$ as

$$f^{(1)}(\eta) = S_0\left[1 - 2e^{-S_0\eta}\,\frac{M'_{k,\frac{\mu}{2}}\left(2\sqrt{b_{11}}e^{-S_0\eta}\right) + \lambda W'_{k,\frac{\mu}{2}}\left(2\sqrt{b_{11}}e^{-S_0\eta}\right)}{M_{k,\frac{\mu}{2}}\left(2\sqrt{b_{11}}e^{-S_0\eta}\right) + \lambda W_{k,\frac{\mu}{2}}\left(2\sqrt{b_{11}}e^{-S_0\eta}\right)}\right] \tag{3.19a}$$

where

$$\lambda = \frac{\left\{\frac{1}{2} - \left(2\sqrt{b_{11}} - k\right)\right\} M_{k,\frac{\mu}{2}}\left(2\sqrt{b_{11}}\right) - \left(\frac{1}{2} + \frac{\mu}{2} + k\right) M_{k+1,\frac{\mu}{2}}\left(2\sqrt{b_{11}}\right)}{\left\{\left(2\sqrt{b_{11}} - k\right) - \frac{1}{2}\right\} W_{k,\frac{\mu}{2}}\left(2\sqrt{b_{11}}\right) - W_{k+1,\frac{\mu}{2}}\left(2\sqrt{b_{11}}\right)}$$

and $$b_{11} = \frac{\left(3 + k_1^* S_0^2\right)}{8S_0^4}. \tag{3.19b}$$

The expression for velocity function $f_\eta^{(1)}(\eta)$ is obtained as

$$f_\eta^{(1)}(\eta) = 2S_0^2\left[\begin{array}{l} e^{-S_0\eta}\,\dfrac{M'_{k,\frac{\mu}{2}}\left(2\sqrt{b_{11}}e^{-S_0\eta}\right) + \lambda W'_{k,\frac{\mu}{2}}\left(2\sqrt{b_{11}}e^{-S_0\eta}\right)}{M_{k,\frac{\mu}{2}}\left(2\sqrt{b_{11}}e^{-S_0\eta}\right) + \lambda W_{k,\frac{\mu}{2}}\left(2\sqrt{b_{11}}e^{-S_0\eta}\right)} + \\ -e^{-2S_0\eta}\left(\dfrac{M'_{k,\frac{\mu}{2}}\left(2\sqrt{b_{11}}e^{-S_0\eta}\right) + \lambda W'_{k,\frac{\mu}{2}}\left(2\sqrt{b_{11}}e^{-S_0\eta}\right)}{M_{k,\frac{\mu}{2}}\left(2\sqrt{b_{11}}e^{-S_0\eta}\right) + \lambda W_{k,\frac{\mu}{2}}\left(2\sqrt{b_{11}}e^{-S_0\eta}\right)}\right)^2 + \\ + \left\{-\dfrac{1}{4} + a_{11} + b_{11}e^{-2S_0\eta} + c_{11}e^{-S_0\eta}\right\} \end{array}\right] \tag{3.20a}$$

where

$$a_{11} = \left[1 - \frac{\left(3 + k_1^* S_0^2\right)}{4S_0^2} - \frac{k_1^*}{2}\right]\frac{1}{2S_0^2},$$

$$c_{11} = \frac{k_1^*}{4S_0^2}. \tag{3.20b}$$

The expressions for b_{11} and λ are given by Eq.(3.19b). The functions $M'_{k,\mu}(z)W'_{k,\mu}(z)$ are the derivatives of the Whitaker functions $M_{k,\mu}(z)$ and $W_{k,\mu}(z)$ respectively and they are defined as

$$M'_{k,\frac{\mu}{2}}(z)=\left[\left(\frac{z}{2}-k\right)M_{k,\frac{\mu}{2}}(z)+\left(\frac{1}{2}+\frac{\mu}{2}+k\right)M_{k+1,\frac{\mu}{2}}(z)\right]\frac{1}{z},$$

(3.20c)

$$W'_{k,\frac{\mu}{2}}(z)=\left[\left(\frac{z}{2}-k\right)W_{k,\frac{\mu}{2}}(z)-W_{k+1,\frac{\mu}{2}}(z)\right]\frac{1}{z}.$$

The dimensionless skin-friction coefficient C_f is expressed as

$$C_f=-\frac{\left(\gamma\frac{\partial u}{\partial y}-k_0\left\{u\frac{\partial^2 u}{\partial x\partial y}-2\frac{\partial u}{\partial y}\frac{\partial v}{\partial y}\right\}\right)}{U_0^2\exp\left(\frac{2x}{l}\right)} \qquad \text{at} \qquad y=0,$$

or (3.21)

$$C_f=-\frac{1}{\sqrt{2\mathrm{Re}}}f^{(1)}_{\eta\eta}\left[1-\frac{7}{2}k_1^*\right]=\frac{\left(3+2k_1^*S_0^2\right)}{2S_0}\frac{1}{\sqrt{2\mathrm{Re}}}\left[1-\frac{7}{2}k_1^*\right].$$

Here, $\mathrm{Re}=\frac{U_w l}{\gamma}$ is the non-dimensional Reynolds number.

4. Verification of the solution

Solutions derived in the previous section may be verified at the boundary sheet. Substituting the expression of λ given the Eq.(3.19b) in Eq.(3.19a) and making $\eta=0$ therein it can be easily seen that the boundary condition $f^{(1)}(\eta)=0$ at $\eta=0$ is satisfied. Now, in order to verify the solution at the second boundary condition $f^{(1)}_\eta(\eta)=1$ at $\eta=0$ we make use of the relations (3.19b) and (3.20c) and we obtain

$$\frac{M'_{k,\frac{\mu}{2}}\left(2\sqrt{b_{11}}e^{-S_0\eta}\right)+\lambda W'_{k,\frac{\mu}{2}}\left(2\sqrt{b_{11}}e^{-S_0\eta}\right)}{M_{k,\frac{\mu}{2}}\left(2\sqrt{b_{11}}e^{-S_0\eta}\right)+\lambda W_{k,\frac{\mu}{2}}\left(2\sqrt{b_{11}}e^{-S_0\eta}\right)}=\frac{P_1+P_2+P_3}{e^{-S_0\eta}[Q_1+Q_2]} \tag{4.1}$$

where

$$P_1=-\left(\sqrt{b_{11}}e^{-S_0\eta}-k\right)M_{k,\frac{\mu}{2}}\left(2\sqrt{b_{11}}e^{-S_0\eta}\right)W_{k+1,\frac{\mu}{2}}\left(2\sqrt{b_{11}}\right)+$$
$$-\left\{\frac{1}{2}-\left(\sqrt{b_{11}}-k\right)\right\}M_{k,\frac{\mu}{2}}\left(2\sqrt{b_{11}}\right)W_{k+1,\frac{\mu}{2}}\left(2\sqrt{b_{11}}e^{-S_0\eta}\right), \tag{4.1a}$$

$$P_2=\left\{\frac{1}{2}-\sqrt{b_{11}}-k\right\}\left(\sqrt{b_{11}}e^{-S_0\eta}-k\right)\begin{pmatrix}M_{k,\frac{\mu}{2}}\left(2\sqrt{b_{11}}\right)W_{k,\frac{\mu}{2}}\left(2\sqrt{b_{11}}e^{-S_0\eta}\right)+\\-M_{k,\frac{\mu}{2}}\left(2\sqrt{b_{11}}e^{-S_0\eta}\right)W_{k,\frac{\mu}{2}}\left(2\sqrt{b_{11}}\right)\end{pmatrix}+$$

$$+\left\{\frac{1}{2}+\frac{\mu}{2}+k\right\}\begin{pmatrix}M_{k+1,\frac{\mu}{2}}\left(2\sqrt{b_{11}}\right)W_{k+1,\frac{\mu}{2}}\left(2\sqrt{b_{11}}e^{-S_0\eta}\right)+\\-M_{k+1,\frac{\mu}{2}}\left(2\sqrt{b_{11}}e^{-S_0\eta}\right)W_{k+1,\frac{\mu}{2}}\left(2\sqrt{b_{11}}\right)\end{pmatrix},$$

$$P_3=-\left\{\frac{1}{2}+\frac{\mu}{2}+k\right\}\left\{\frac{1}{2}+\sqrt{b_{11}}\left(e^{-S_0\eta}-1\right)\right\}M_{k+1,\frac{\mu}{2}}\left(2\sqrt{b_{11}}\right)W_{k+1,\frac{\mu}{2}}\left(2\sqrt{b_{11}}e^{-S_0\eta}\right), \qquad \text{c.d. (4.1a)}$$

$$Q_1=-M_{k,\frac{\mu}{2}}\left(2\sqrt{b_{11}}e^{-S_0\eta}\right)W_{k+1,\frac{\mu}{2}}\left(2\sqrt{b_{11}}\right)+$$

$$-\left\{\frac{1}{2}-\frac{\mu}{2}+k\right\}M_{k+1,\frac{\mu}{2}}\left(2\sqrt{b_{11}}\right)W_{k,\frac{\mu}{2}}\left(2\sqrt{b_{11}}e^{-S_0\eta}\right),$$

$$Q_2=\left(\frac{1}{2}-\left(\sqrt{b_{11}}-k\right)\right)\begin{Bmatrix}M_{k,\frac{\mu}{2}}\left(2\sqrt{b_{11}}\right)W_{k,\frac{\mu}{2}}\left(2\sqrt{b_{11}}e^{-S_0\eta}\right)+\\-M_{k,\frac{\mu}{2}}\left(2\sqrt{b_{11}}e^{-S_0\eta}\right)W_{k,\frac{\mu}{2}}\left(2\sqrt{b_{11}}\right)\end{Bmatrix}.$$

Substituting $\eta=0$ in Eqs (4.1) and (4.1a) we obtain

$$\frac{M'_{k,\frac{\mu}{2}}\left(2\sqrt{b_{11}}e^{-S_0\eta}\right)+\lambda W'_{k,\frac{\mu}{2}}\left(2\sqrt{b_{11}}e^{-S_0\eta}\right)}{M_{k,\frac{\mu}{2}}\left(2\sqrt{b_{11}}e^{-S_0\eta}\right)+\lambda W_{k,\frac{\mu}{2}}\left(2\sqrt{b_{11}}e^{-S_0\eta}\right)}=\frac{1}{2} \qquad \text{at} \qquad \eta=0. \tag{4.2}$$

Substituting $\eta=0$ in Eq.(3.20a) and noting that

$$a_{11}+b_{11}+c_{11}=\frac{1}{2S_0^2}, \tag{4.3}$$

and also making use of Eq.(4.2) it can be easily verified that $f_\eta^{(1)}(\eta)=1$ at $\eta=0$. Hence, the first-order solution satisfies the second boundary condition at $\eta=0$.

The accuracy of the zero-order and first-order solutions obtained by Eqs (3.11) and (3.19) respectively may also be verified with the numerical solutions of Eq.(3.4) by employing the Runge-Kutta fourth order method. It is seen that the profiles of zero-order and first-order solutions are very close to the profile of the numerical solution of the non linear Eq.(3.4) obtained by employing the Runge-Kutta fourth order method with shooting in the region very near the boundary (Fig.2).

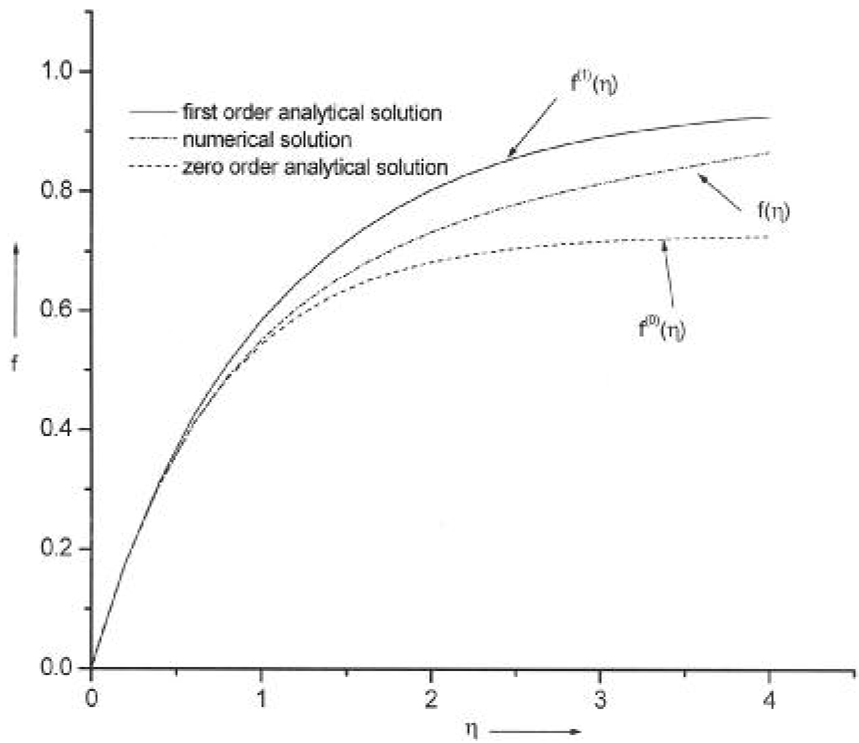

Fig.2. Profiles of $f(\eta)$ obtained from zero order, first order and numerical solutions when $k_1^* = 0.2$.

5. Results and discussion

Approximate analytical similarity solutions (zero and first order) of the highly non-linear boundary layer equation of viscoelastic fluid flow over an exponentially stretching impermeable sheet have been obtained. In the process of deriving mathematical solutions, the highly non-linear partial differential equation characterizing the flow has been converted into a non-linear ordinary differential equation by applying suitable similarity transformations. Sequential similarity solutions of the transformed momentum equation are obtained by solving the non-linear Riccati type equation analytically. The zero-order solution for the dimensionless stream function has been obtained analytically which satisfies all the boundary conditions. The first-order solution of f has also been derived analytically in the form of the confluent hypergeometric Whittaker's function. These solutions may be verified with the numerical solution of Eq.(3.4) by employing the Runge-Kutta fourth order method (Fig.2). In order to solve the fourth order non-linear differential Eq.(3.4) numerically it is necessary to have four boundary conditions. However, there are three boundary conditions prescribed by Eq.(3.5). Hence, we generate the fourth boundary condition by substituting the first two boundary conditions of Eq.(3.5) in Eq.(3.4) in the following form

$$f_{\eta\eta\eta}(0) = \frac{4 - 3k_1^* f_{\eta\eta}^2(0)}{2(1 - 3k_1^*)}. \tag{5.1}$$

The graphical analysis of Fig.2. reveals that the profiles of zero and first-order solutions are very close to the profile of the numerical solution of the non-linear Eq.(3.4). Also, the numerical solution of Eq.(3.4) for $f_\eta(\eta)$, using the Runge-Kutta fourth order method with shooting, has been obtained which matches very well with the analytical solutions of zero-order $f_\eta^{(0)}(\eta)$ and first-order $f_\eta^{(1)}(\eta)$ in the region which is very close to the boundary sheet (Fig.3). It is important to note that all these solutions involve an exponential dependence of (i) the similarity variable η (ii) the stream function $f(\eta)$ and (iii) the velocity component $f_\eta(\eta)$ on the coordinate *x* along the direction of stretching.

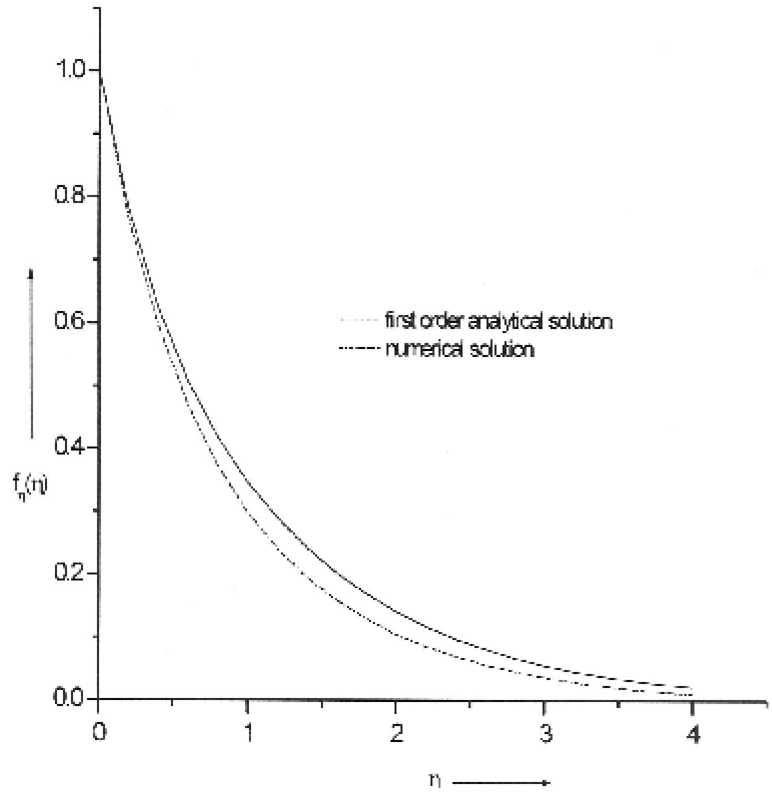

Fig.3. Velocity profile $f_\eta(\eta)$ obtained from first order solution and numerical solution when $k_1^* = 0.1$.

In order to gain some knowledge on the qualitative behaviour of the flow characteristics, the numerical values of the results are plotted graphically for a typical choice of physical parameters in Figs 4-5. The graphs for the non-dimensional velocity profile $f_\eta^{(1)}(\eta)$ for different values of the viscoelastic parameter k_1^* are shown in Fig.4. The analysis of the figure demonstrates that the effect of the viscoelastic parameter k_1^* is to decrease velocity throughout the boundary layer flow field, which is quite obvious. The graphs of the non-dimensional skin-friction parameter C_f vs. viscoelastic parameter k_1^* for different values of the Reynolds number Re are shown in Fig.5. From this figure we notice that the increase of non-dimensional viscoelastic parameter k_1^* leads to the decrease of skin-friction parameter C_f. This result is the consequence of the fact that the elastic property in a viscoelastic fluid reduces the frictional force. These results may have great significance in polymer proceeding industry, as the choice of higher order viscoelastic fluid would reduce the power consumption for stretching the boundary sheet. The effect of the Reynolds number on the skin-friction coefficient is also seen to reduce the skin-friction coefficient C_f as a reduction of viscous property of the fluid results in the decrease of frictional force or drag force.

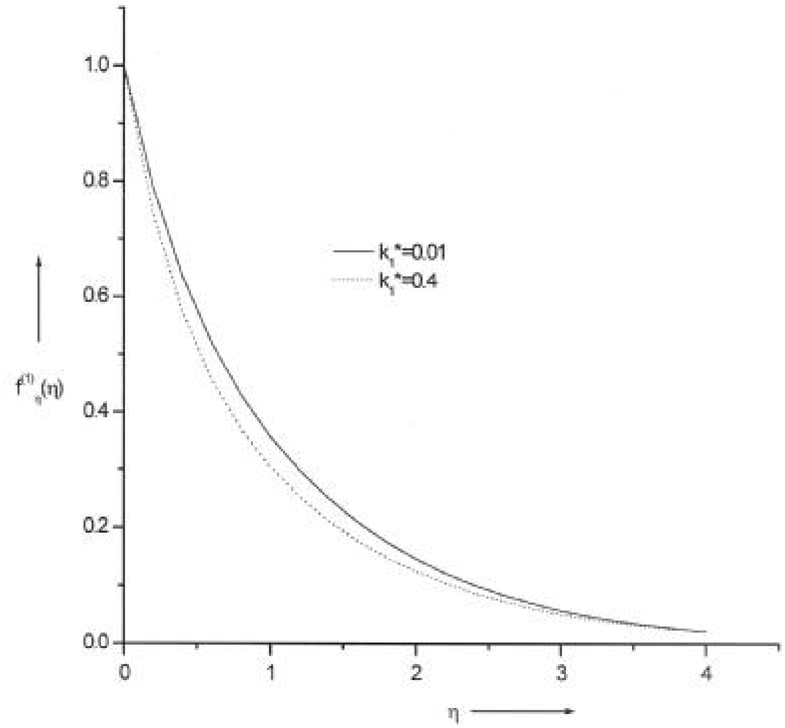

Fig.4. Velocity profile $f_\eta^{(1)}(\eta)$ for different values of k_1^*.

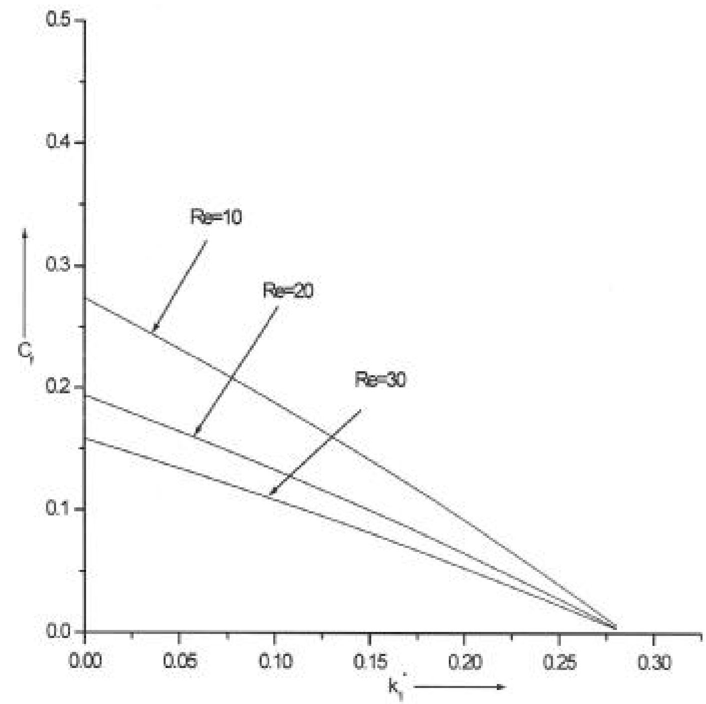

Fig.5. Graph of skin friction parameter C_f vs viscoelastic parameter k_1^* for different values of Reynolds number Re.

6. Conclusions

A mathematical problem has been formulated for the momentum transfer in a viscoelastic fluid flow over an exponentially stretching impermeable sheet. In the solution procedure a highly non-linear differential equation is converted into an ordinary differential equation by applying similarity transformations. Sequential similarity solutions of the transformed momentum equation are obtained analytically by solving the non-linear Riccati type equation repeatedly. The first-order approximate solutions for the stream function $f(\eta)$ and the velocity function $f_\eta(\eta)$ are obtained in the form of the confluent hypergeometric Whittaker functions. The solutions are verified at the boundary sheet. These solutions are also compared with the numerical solution of the problem obtained by employing the Runge-Kutta fourth order method with shooting and good accuracy has been found near the region very close to the boundary sheet. The zero-order velocity profile $f_\eta^{(0)}(\eta)$ and first-order velocity profile $f_\eta^{(1)}(\eta)$ are also compared with the numerical solutions obtained by employing the Runge-Kutta fourth order method with shooting and the desired accuracy has been achieved. Expressions are also obtained for the dimensionless skin-friction coefficient C_f. The derived solutions involve an exponential dependence of (i) the similarity variable η (ii) the stream function $f(\eta)$ and (iii) the velocity profile $f_\eta(\eta)$ on the flow directional coordinate.

The important findings of the graphical analysis of the results of the present problem are as follows.

1. Both zero-order and first-order solutions of the stream function $f(\eta)$ have good accuracy near the region very close to the boundary.
2. Both zero-order and first-order solutions of the velocity function $f_\eta(\eta)$ have good accuracy near the region very close to the boundary.
3. The effect of increasing the values of the viscoelastic parameter k_1^* is to decrease the velocity throughout the boundary layer.
4. The effect of increasing the values of the viscoelastic parameter k_1^* is to decrease the skin-friction parameter C_f and the effect of increasing values of the Reynolds number Re is also to decrease the skin-friction coefficient C_f.
5. The limiting cases of the results of the paper when $k_1^* \to 0$ are in excellent agreement with the results of Elbashbeshy (2001).

Acknowledgment

The author is thankful to Prof. B.N. Mandal, Indian Statistical Institute, Kolkata for valuable discussion. The financial support extended by University Grants Commission, New Delhi, under Special Assistance Programme DRS is greatly acknowledged.

Nomenclature

A_1 and A_2 – kinematical tensors
C_f – dimensionless skin-friction coefficient
f – dimensionless stream function
k_0 – elastic parameter
k_1^* – viscoelastic parameter
l – reference length
$M(a,b,z)$ – Kummer's function

$M_{k,\mu}(z)$ and $W_{k,\mu}(z)$ – the Whittaker's functions
p – pressure
Re – non-dimensional Reynolds number
T – stress tensor
u,v – velocity components
U_0 – constant
γ – kinematics coefficient of viscosity
η – similarity variable
μ – dynamic viscosity
$\mu\ \alpha_1, \alpha_2$ – the normal stress moduli
$\psi(x, y)$ – stream function

References

Abramowitz M. and Stegun I.A. (1970): *Handbook of Mathematical functions*. – New York: Dover Publications, Inc.

Ali M.E. (195): *On thermal boundary layer on a power-law stretched surface with suction or injection.* – Int. J. Heat Mass Flow, vol.16, pp.280-290.

Andersson H.I. (1992): *MHD flow of a viscoelastic fluid past a stretching surface*. – Acta. Mech., vol.95, pp.227-230.

Beard D.W. and Walters K. (1964): *Elastico-viscous boundary layer flows. 1. Two dimensional flow near a stagnation point*. – Proc. Comb. Phil. Soc., vol.60, pp.667.

Char M.I. (1994): *Heat and mass transfer in a hydromagnetic flow of viscoelastic fluid over a stretching sheet*. – J. of Math. Analysis and Appl., vol.186, pp.674-689.

Coleman B.D. and Noll W. (1960): *An approximation theorem for functionals with applications in continuum mechanics*. – Arch. Rat. Mech. Anal., vol.6, pp.355.

Dandapat B.S. and Gupta A.S. (1989): *Flow and heat transfer in a viscoelastic fluid over a stretching sheet*. – Int. J. Non-linear Mech., vol.24, No.3, pp.215-219.

Elbashbeshy E.M.A. (2001): *Heat transfer over an exponentially stretching continuous surface with suction*. – Arch. Mech., vol.53, No.6, pp.643-651.

Fosdick R.L and Rajagopal K.R. (1979): *Anomalous features in the model of second order fluids*. – Arch. Ration. Mech. Anal., vol.70, pp.145.

Gupta P.S. and Gupta A.S. (1977): *Heat and mass transfer on a stretching sheet with suction or blowing*. – Canad. J. of Chem. Eng., vol.55, pp.744-746.

Kumaran V. and Ramanaiah G. (1996): *A note on the flow over a stretching sheet*. – Acta Mech., vol.116, pp.229-233.

Lawrence P.S. and Rao B.N. (1992): *Heat transfer in the flow of viscoelastic fluid over a stretching sheet*. – Acta. Mech., vol.93, pp.53-61.

Mcleod J.B. and Rajagopal K.R. (1987): *On the uniqueness of the flow of a Navier-Stokes fluid due to a stretching boundary*. – Arch. Rational Mech. Anal., vol.98, pp.385-393.

Rajagopal K.R., Na T.Y. and Gupta A.S. (1984): *Flow of a viscoelastic fluid over a stretching sheet*. – Rheol. Acta., vol.23, pp.213-215.

Rajagopal K.R., Na T.Y. and Gupta A.S. (1987): *A non similar boundary layer on a stretching sheet in a non-Newtonian fluid with uniform free stream*. – J. Math. Phy. Sc., vol.21, No.2, pp.189-200.

Rao B.N. (1996): *Technical note: Flow of a fluid of second grade over a stretching sheet*. – Int. J. Non-Linear Mech. vol.31, No.4, pp.547-550.

Rollins D. and Vajravelu K. (1991): *Heat transfer in a second order fluid over a continuous stretching surface*. – Acta. Mech., vol.89, pp.167-178.

Sakiadis B.C. (1961): *Boundary layer behaviour on continuous solid surfaces: 1. Boundary layer equations for two-dimensional and axisymmetric flow*. – A.I.Ch. E.J., vol.7, pp.26-28.

Sonth R.M., Sujit Kumar Khan, Abel M.S. and Prasad K.V. (2002): *Heat and mass transfer in a viscoealatic fluid flow over an accelerating surface with heat source/sink and viscous dissipation*. – Heat and Mass Transfer, vol.38, pp.213-220.

Subhash Abel M., Sujit Kumar Khan and Prasad K.V. (2002): *Study of viscoelastic fluid flow and heat transfer over a stretching sheet with variable viscosity*. – International J. Non-Linear Mechanics, vol.37, pp.81-88.

Troy W.C., Overman E.A., Ermentrout H.G.B. and Keener J.P. (1987): *Uniqueness of flow of a second order fluid past a stretching sheet*. – Qurat. Appl. Math., vol.44, pp.753-755.

Received: August 18, 2005

Revised : October 2, 2005