THE STUDY OF PRESSURE DROPS IN THE FLOW OF A POLYMER MODELED AS A BINGHAM FLUID IN CONICAL CHANNELS

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Technologies applied in polymers processing are permanently improved due to updating the knowledge on material proprieties, processes and phenomena during the processing. To determine the pressure distribution, one should define the geometrical shape of a channel, in the flow will be held the flow. The aim of calculations carried out is to determine the possibility of mathematical modelling of polymer flows between conical parallel surfaces. In this work, the flow of a polymer in conical channels was considered. To describe a melted polymer the model of a viscoplastic fluid was used but final results were illustrated by a flow of the Bingham fluid. The Bingham fluid chosen to modelling the flow of a polymer may by considered as legitimate, because its use will allow us to illustrate analytical methods of calculations. For the flow configuration and model under consideration the geometrical sizes of the channel and material coefficients of the fluid will be chosen on the basis of experimental data contained in literature. In the article, the on defining dimensionless pressure distribution for the flow in a conical channel as well as in a conical annular channel with the influence of inertia were presented. The results of calculations were introduced in tabular forms as well as in graphic forms.

Key words: conical channels, polymer, viscoplastic fluid, Bingham fluid.

1. Introduction

Convergent flows of viscous fluids have received considerable attention in recent years for a variety of reasons and several distinct approaches to the problem have been used. Such flows occur frequently in many industrial applications, particulary in manufacturing processes involving polymer melts and in those instances the prediction of pressure losses is often of paramount concern.

Manufacturing is often realized by extrusion where a molten polymer forms a mixture of liquid polymer with solidified particles. The flow of this mixture may be modeled as the flow of viscoplastic fluids (Avenas ***et al.***, 1982; Binding, 1988; Cogswell, 1972, 1978). To describe the rheological behaviour of such fluids the non-linear model of Shulman is often used (Shulman, 1975). Sometimes viscoplastic fluids may include some polymer liquid cristals and some filled thermoplastics and they can manifest stresses that develop orthogonal to planes of shear which is typical to viscoelastic fluids.

To obtain a model that does exhibit both viscoplastic and viscoelastic behaviour we propose the following two constitutive equations (Walicka, 2002a, 2002b; Walicki, Walicka, 1999)

– model *I*: $$\boldsymbol{T} = -p\boldsymbol{I} + M\boldsymbol{A}_1 + \alpha_1 \boldsymbol{A}_1^2 + \beta_1 \boldsymbol{A}_2 , \tag{1.1}$$

– model *II*: $$\boldsymbol{T} = -p\boldsymbol{I} + M\left(\boldsymbol{A}_1 + \alpha_2 \boldsymbol{A}_1^2 + \beta_2 \boldsymbol{A}_2\right) \tag{1.2}$$

where p is the pressure , α_i and β_i $(i = 1 \text{ or } 2)$ are the material moduli, $\boldsymbol{I}$ is the unit tensor, $\boldsymbol{A}_1$ and $\boldsymbol{A}_2$ are the first two Rivlin-Ericksen tensors defined by (Rivlin and Ericksen, 1955)

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$$\boldsymbol{A}_1 = \boldsymbol{L} + \boldsymbol{L}^T, \qquad \boldsymbol{A}_2 = \dot{\boldsymbol{A}}_1 + \boldsymbol{A}_1\boldsymbol{L} + \boldsymbol{L}^T\boldsymbol{A}_1, \qquad \boldsymbol{L} = grad\,\mathbf{v} \tag{1.3}$$

here $\mathbf{v}$ is the velocity vector and $(\dot{\ })$ represents the material derivative with respect to time. The coefficient M represents the Shulman function of viscoplasticity which is given by the following formulae (Shulman, 1975)

$$M = \left[\tau_0^{\frac{1}{n}} + (\mu A)^{\frac{1}{m}}\right]^n A^{-1}, \qquad A = \left[\frac{1}{2}tr\left(\boldsymbol{A}_1^2\right)\right]^{\frac{1}{2}} \tag{1.4}$$

where τ_0 is the yield shear stress, μ is the coefficient of plastic viscosity, m and n are non-linearity indices, A is the second invariant of the stretching tensor.

Following the considerations suggested in their earlier works the authors present here an approximate analysis for a convergent flow of generalized second grade fluids. The detailed formulae for pressure drops is given for a Bingham type fluids $(\tau_0 \neq 0, m = n = 1)$.

2. Pressure drop in conical die

The flow configuration is presented in Fig.1. There are three basic modes of deformation: bulk deformation, simple shear deformation and simple tension; we also have the effect of normal stresses. For the purpose of this analysis we assume that the bulk deformations are sufficiently small to be neglected.

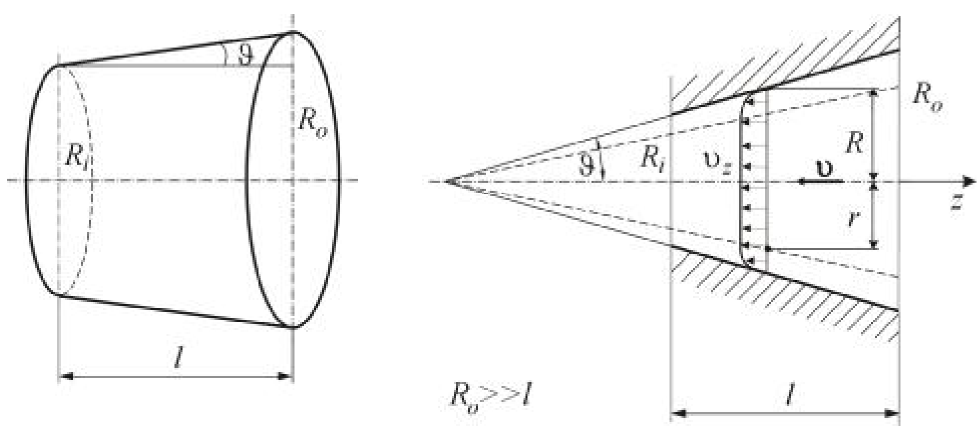


Fig.1. Schematic diagram of a conical flow.

The corresponding velocity component is υ_z which is parallel to the axis of z. Considering an equilibrium of a convergent element of length dz (Fig.2) we have (Michalski, 2004; Walicka, 2002a, 2002b; Walicki, Walicka, 1999):

- for the pressure drops due to shear flow:

$$\Delta p_s = -\operatorname{ctg}\vartheta \int_{R_i}^{R_o} \frac{\partial p}{\partial z} dR \qquad \text{or} \qquad \Delta p_s = 2\tau_0 \operatorname{ctg}\vartheta \int_{R_i}^{R_o} \frac{Y dR}{R} \tag{2.1}$$

where Y is the function connected with the flow rate Q. The value of Y for the Bingham fluid is given by the solution to the following equation

$$3Y^4 - 4Y^3(1+3K)+1=0\,. \tag{2.2}$$

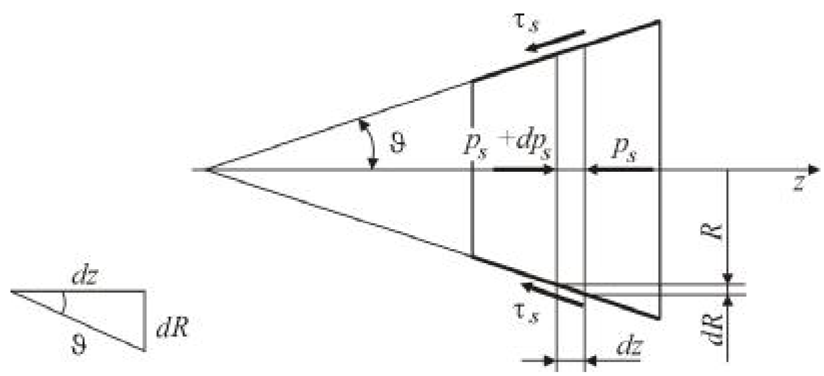


Fig.2. Schematic diagram of forces for the analysis of the pressure drop due to telescopic shear.

- for the pressure drops due to extensional flow (schematic diagram of forces for the analysis of the pressure drop due to extensional flow is shown in Fig.3.)

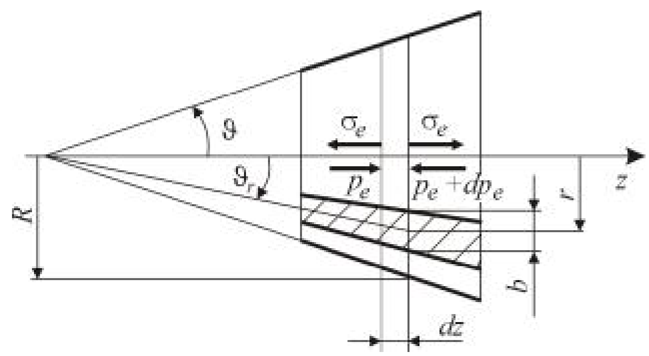


Fig.3. Schematic diagram of forces for the analysis of the pressure drop due to extensional flow.

$$\Delta p_e = 2\int_{R_i}^{R_o} \sigma_e \frac{dR}{R} \tag{2.3a}$$

where σ_e is given by

$$\sigma_e = \frac{2\tau_0\,\mathrm{tg}\,\vartheta}{R^4}\int_0^R Yr^3 dr \tag{2.3b}$$

- for the pressure drops due to normal stresses (schematic diagram of forces for the analysis of the pressure drop due to normal stresses is shown in Fig.4.)

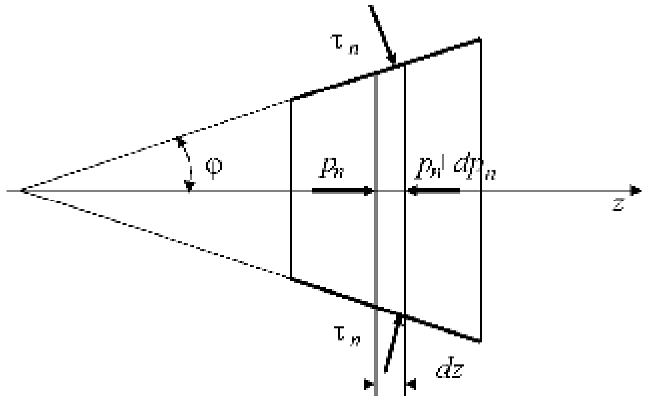


Fig.4. Schematic diagram of forces for the analysis of the pressure drop due to normal stresses.

$$\Delta p_n = 2\int_{R_i}^{R_0} \tau_n \frac{dR}{R}. \tag{2.4}$$

To find τ_n consider the stress equation of motion in the r-direction

$$\frac{\partial p}{\partial r} = \frac{1}{r}\frac{d}{dr}\left(r\tau_{rr}\right). \tag{2.5}$$

Integrating this equation with the boundary condition $\tau_{rr} = 0$ for $r = 0$ we obtain

$$\tau_{rr} = \frac{1}{r}\int r\frac{\partial p}{\partial r}dr, \tag{2.6}$$

but on the die wall there is

$$\tau_{rr}\big|_{r=R} = \tau_n = \Phi(R) \tag{2.7}$$

where $\Phi(R)$ is a known function of R (Walicka, 2002a, 2002b).

3. Example of application

For the Bingham type fluid one has

$$M = \tau_{rz}\left(-\frac{d\upsilon_z}{dr}\right)^{-1} = \mu\frac{\frac{r}{R}Y}{\frac{r}{R}Y - 1}, \qquad \sigma_e = \frac{\tau_0 Y \tan\vartheta}{2},$$

$$\Phi_1(R) = \left(\alpha_1 + 2\beta_1\right)\frac{\tau_0^2}{\mu^2}(Y-1)^2, \qquad \Phi_2(R) = \left(\alpha_2 + 2\beta_2\right)\frac{\tau_0^2}{\mu}Y(Y-1).$$

The value of Y is given by the solutions to Eq.(2.2); its solutions for a large value of $K = K_l$ and for a small value of $K = K_s$ are, respectively

$$Y_l = 4K_l + \frac{4}{3}, \qquad Y_s = 1 + \sqrt{2K_s} \qquad \text{where} \qquad K = \frac{\mu Q}{\pi R^3 \tau_0}.$$

The pressure drops are as follows

$$\Delta p_s = \frac{2\tau_0 \cot\vartheta}{3} p_\upsilon, \qquad \Delta p_e = \frac{\tau_0 \tan\vartheta}{3} p_\upsilon, \tag{3.1}$$

and

$$\Delta p_{n1} = \frac{(\alpha_1 + 2\beta_1)\tau_0^2}{3\mu^2} p_{l1}, \qquad \Delta p_{n2} = \frac{(\alpha_2 + 2\beta_2)\tau_0^2}{\mu} p_{l2}, \tag{3.2}$$

or

$$\Delta p_{n1} = \frac{(\alpha_1 + 2\beta_1)\tau_0^2}{3\mu^2} p_{s1}, \qquad \Delta p_{n2} = \frac{(\alpha_2 + 2\beta_2)\tau_0^2}{\mu} p_{s2} \tag{3.3}$$

where for a large K

$$\begin{aligned} p_\upsilon &= 4K_{li}\left(1-\beta^3\right) - 4\ln\beta, \\ p_{l1} &= 16K_{li}^2\left(1-\beta^6\right) + \frac{16}{3}K_{li}\left(1-\beta^3\right) - \frac{1}{3}\ln\beta, \\ p_{l2} &= 16K_{li}^2\left(1-\beta^6\right) + \frac{40}{3}K_{li}\left(1-\beta^3\right) - \frac{8}{3}\ln\beta, \end{aligned} \tag{3.4}$$

and for a small K

$$\begin{aligned} p_\upsilon &= 2\sqrt{2K_{si}}\left(1-\beta^{3/2}\right) - 3\ln\beta, \qquad p_{s1} = 4K_{si}\left(1-\beta^3\right), \\ p_{s2} &= 4\left[\sqrt{2K_{si}}\left(1-\beta^{3/2}\right) + 2K_{si}\left(1-\beta^3\right)\right], \end{aligned} \tag{3.5}$$

here $K_i = K\big|_{R=R_i}$. The above formulae may be rewritten as

$$\Delta p_{1,2} = \zeta_{1,2}\frac{\rho\upsilon_i^2}{2} \qquad \text{where} \qquad \upsilon_i = \frac{Q}{\pi R_i^2}, \tag{3.6}$$

and ζ_1 or ζ_2 are the dimensionless coefficients of local pressure drop, ρ - the fluid density, υ_i - the velocity in the R_i cross-section of the die. Then the coefficients ζ_1 and ζ_2 are equal to

$$\zeta_1 = \zeta_{Bl} + \gamma_{l1}, \qquad \zeta_2 = \zeta_{Bl} + \gamma_{l2}, \tag{3.7}$$

for large K and

$$\zeta_1 = \zeta_{Bs} + \gamma_{s1}, \qquad \zeta_2 = \zeta_{Bs} + \gamma_{s2}, \tag{3.8}$$

for small K ; here

$$\zeta_{Bl} = \frac{16}{3Re}(2\cot\vartheta + \tan\vartheta)\left(1 - \beta^3 - \frac{1}{2}\mathrm{Bg}\ln\beta\right),$$

$$\zeta_{Bs} = \frac{8(\mathrm{Bg})^{1/2}}{3Re}(2\cot\vartheta + \tan\vartheta)\left(1 - \beta^{3/2} - \frac{3}{4}(\mathrm{Bg})^{1/2}\ln\beta\right),$$

$$\gamma_{l1} = \frac{\alpha_1 + 2\beta_1}{\rho R_i^2}\left[\frac{32}{3}\left(1-\beta^6\right) + \frac{16}{9}\mathrm{Bg}\left(1-\beta^3\right) - \frac{1}{18}(\mathrm{Bg})^2\ln\beta\right], \tag{3.9}$$

$$\gamma_{l2} = \frac{(\alpha_2 + 2\beta_2)\ \mu}{\rho R_i^2}\left[\frac{32}{3}\left(1-\beta^6\right) + \frac{40}{9}\mathrm{Bg}\left(1-\beta^3\right) - \frac{4}{3}(\mathrm{Bg})^2\ln\beta\right],$$

$$\gamma_{s1} = \frac{\alpha_1 + 2\beta_1}{\rho R_i^2}\ \frac{4}{3}\mathrm{Bg}\left(1-\beta^3\right),$$

$$\gamma_{s2} = \frac{(\alpha_2 + 2\beta_2)\ \mu}{\rho R_i^2}\left[\frac{4}{3}(\mathrm{Bg})^{3/2}\left(1-\beta^{3/2}\right) + \frac{8}{3}\mathrm{Bg}\left(1-\beta^3\right)\right].$$

The Reynolds and Bingham numbers are equal, respectively, to

$$\mathrm{Re}(m) = \frac{\rho\upsilon_i^{2-\frac{1}{m}}(2Ri)^{\frac{1}{m}}}{\mu^{\frac{1}{m}}}, \qquad \mathrm{Re} = \mathrm{Re}(1); \qquad \mathrm{Bg} = \frac{2R_i\tau_0}{\mu\upsilon_i}. \tag{3.10}$$

In polymer melts processing the following formula for ζ is frequently accepted (Avenas *et al.*, 1982)

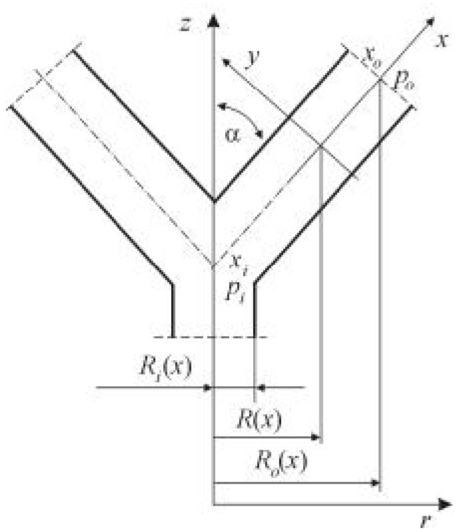
$$\zeta = \frac{32\cot\vartheta}{3\mathrm{Re}}\left(1-\beta^3\right), \tag{3.11}$$

as the first approximation for the pressure drop in the die. It is clear that this formula represents only the pressure drop due to the shear flow of a Newtonian fluid.

Note that there is $\mathrm{Bg}\cdot K_i = 2$; for a large K_i the value of Bg is small and the right sides of the third and fourth term from formulae (3.9) may limit to the first term in square brackets.

4. Conical annular channel – solution to Bingham fluid

Let us consider the flow between conical parallel surfaces (the thickness of the slot is constant), as shown in Fig.5.



$$R(x) = x \sin\alpha$$
$$R_o(x) = x_o \sin\alpha$$
$$R_i(x) = x_i \sin\alpha$$
$$h = const$$

Fig.5. Clearance between conical parallel surfaces.

Assuming that the thickness of the clearance is small the equations of movement can be written in the form (Michalski, 2004; Walicka, 2002a, 2002b)

$$\frac{1}{R}\frac{\partial(R\upsilon_x)}{\partial x} + \frac{\partial \upsilon_y}{\partial y} = 0, \tag{4.1}$$

$$\rho\left(\upsilon_x \frac{\partial \upsilon_x}{\partial x} + \upsilon_y \frac{\partial \upsilon_x}{\partial y}\right) = -\frac{dp}{dx} + \frac{\partial}{\partial y}\left(S\left[\tau_0^{1/n} + \left(\mu\left|\frac{\partial \upsilon_x}{\partial y}\right|\right)^{1/m}\right]^n\right) \tag{4.2}$$

where S is the function of sign

$$S = \mathrm{sgn}\left(\frac{\partial \upsilon_x}{\partial y}\right).$$

Solving the above equations of movement for the flow of a polymer modelled as a Bingham fluid we can describe the pressure distribution in the clearance. Analytic solutions of these equations exist only for large $(K \geq 5)$ as well as for small $\left(K \leq 5 \times 10^{-2}\right)$ values of the de Saint – Venant numbers

$$K = \frac{\mu Q \tau_0^{-m/n}}{4\pi R_o h^2}.$$

According to (Michalski, 2004; Walicka, 2002a, 2002b)

$$p = p_i + [S(x) - S_i] + [T(x) - T_i],$$

or (4.3)

$$p = p_o + [S(x) - S_o] + [T(x) - T_o]$$

where there are:

- for a small K

$$S(x) = -\frac{\tau_0}{h} x - \frac{2\tau_0 (2K_H)^{1/2}}{h^2 \sin^{1/2}\alpha} x^{1/2},$$

$$T(x) = \frac{7\rho\tau_0^2}{180\mu^2} (2K_H)^{3/2} \frac{1}{hx^{3/2}\sin^{3/2}\alpha} \tag{4.4}$$

- for a large K

$$S(x) = -\frac{3\tau_0}{2h} x - \frac{3\tau_0 K_H}{h^3 \sin\alpha} \ln x,$$

$$T(x) = -\frac{\rho\tau_0^2}{120\mu^2} \left[h^2 \ln(x \sin\alpha) + 18(2K_H)^2 \frac{1}{x^2 h^2 \sin^2\alpha} \right]. \tag{4.5}$$

Subtracting formulae (4.3) we obtain the expression for the pressure drop in the clearance

$$\Delta p = p_i - p_o = (S_i - S_o) + (T_i - T_o). \tag{4.6}$$

It may be noticed that if $T(x) = 0$ this expression represents the pressure drop without inertia.

Making suitable calculations we will get the dimensionless pressure distribution in the annular channels:

- for a small K without inertia

$$\tilde{p}_R = \frac{p_R}{p_o} = \frac{p_i}{p_o} - \frac{\tau_0 x_o}{hp_o} \left[\tilde{x} - \varepsilon + \frac{2(2K)^{1/2}}{\sin^{1/2}\alpha} \left(\tilde{x}^{1/2} - \varepsilon^{1/2} \right) \right] \tag{4.7}$$

and taking into account the inertia

$$\tilde{p} = \frac{p}{p_o} = \frac{p_i}{p_o} - \frac{\tau_0 x_o}{hp_o} \left[\tilde{x} - \varepsilon + \frac{2(2K)^{1/2}}{\sin^{1/2}\alpha} \left(\tilde{x}^{1/2} - \varepsilon^{1/2} \right) \right] + \frac{7\rho\tau_0^2 h^2}{90\mu^2 p_o \sin^{3/2}\alpha} (2K)^{3/2} \left(\frac{1}{\tilde{x}^{3/2}} - \frac{1}{\varepsilon^{3/2}} \right). \tag{4.8}$$

- for a large K without inertia

$$\tilde{p}_R = \frac{p_R}{p_o} = \frac{p_i}{p_o} - \frac{3\tau_0 x_o}{2hp_o} \left(\tilde{x} - \varepsilon + \frac{2K}{\sin\alpha} \ln\frac{\tilde{x}}{\varepsilon} \right), \tag{4.9}$$

and taking into account the inertia

$$\tilde{p} = \frac{p}{p_o} = \frac{p_i}{p_o} - \frac{3\tau_0 x_o}{2hp_o}\left(\tilde{x} - \varepsilon + \frac{2K}{\sin\alpha}\ln\frac{\tilde{x}}{\varepsilon} \right) + \frac{\rho\tau_0^2 h^2}{120\mu^2 p_o}\left[\ln\frac{\tilde{x}}{\varepsilon} + \frac{72K^2}{\sin^2\alpha}\left(\frac{1}{\tilde{x}^2} - \frac{1}{\varepsilon^2} \right) \right]. \quad (4.10)$$

Upon the selection of material and flow parameters and then carrying out the calculations we get the results, which are presented in Tab.1.

From the results of analysis and calculations one can observe that the influence of inertia forces on dimensionless pressure distribution is negligibly small in the configuration under consideration. On this basis the following conclusions may be drawn:

- the influence of inertia of a flowing polymer on the pressure distribution in a conical channel between parallel surfaces - for the assumed parameters of flow - is negligibly small,
- to delimit the pressure distribution it suffices practically to apply the formulae not including the inertial terms in equations of motion, which results in a considerable simplification of calculations.

Figures 6 and 7 presents the dimensionless pressure distribution for a flow between conical parallel surfaces without inertia for a Bingham fluid.

Table 1. Specification of results of calculations for the flow between conical parallel surfaces for a Bingham fluid; large and small K with and without inertia.

$\tilde{x}$	Dimensionless pressure distribution $\tilde{p}$ (small K)		Dimensionless pressure distribution $\tilde{p}$ (large K)	
	without inertia	with inertia	without inertia	with inertia
0,1	*8,20835472*	*8,20835472*	*51,47931295*	*51,47931295*
0,2	*7,19654461*	*7,19654457*	*36,41420372*	*36,41419336*
0,3	*6,30156570*	*6,30156565*	*27,56994906*	*27,56993677*
0,4	*5,46706306*	*5,46706301*	*21,27264190*	*21,27262894*
0,5	*4,67120810*	*4,67120804*	*16,37091962*	*16,37090636*
0,6	*3,90280577*	*3,90280572*	*12,35193464*	*12,35192120*
0,7	*3,15520572*	*3,15520566*	*8,94211840*	*8,94210486*
0,8	*2,42407485*	*2,42407480*	*5,97817487*	*5,97816127*
0,9	*1,70640341*	*1,70640335*	*3,35477477*	*3,35476112*
1,0	*1,00000000*	*1,00000000*	*1,00000000*	*1,00000000*

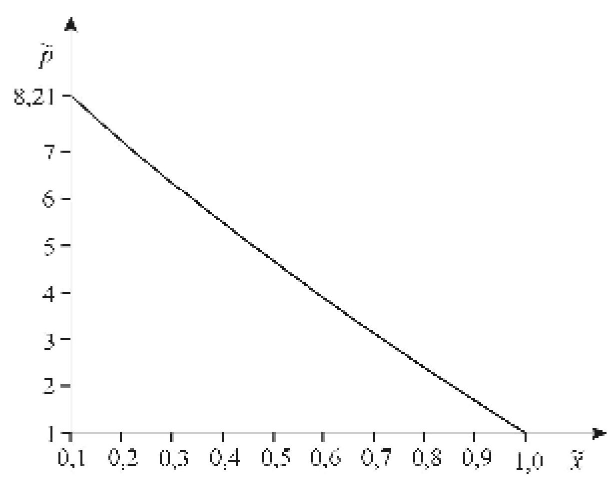


Fig.6. Dimensionless pressure distribution for a Bingham fluid and small K.

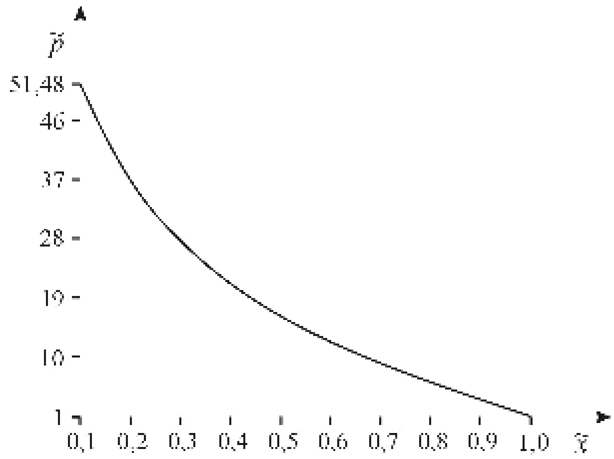


Fig.7. Dimensionless pressure distribution for a Bingham fluid and large K.

Nomenclature

A	– second invariant of the stretching tensor
$\boldsymbol{A_1}$, $\boldsymbol{A_2}$	– first two Rivlin-Ericksen tensors
$\boldsymbol{I}$	– unit tensor
m , n	– non-linearity indices
p	– pressure
$\mathbf{v}$	– velocity vector
υ_i	– velocity in the cross-section of the die
α_i , β_i	– material moduli
μ	– coefficient of plastic viscosity
ρ	– density
τ_0	– yield shear stress
ζ_1 , ζ_2	– dimensionless coefficients of local pressure drop

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