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Brief note

UNSTEADY MHD FLOW DUE TO ECCENTRICALLY ROTATING POROUS DISK AND A THIRD GRADE FLUID AT INFINITY

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This study looks the series solution for the flow of a third grade fluid in a rotating frame. The flow is induced by non-coaxial rotations of porous oscillating disk and a fluid at infinity. It is noted that the obtained expressions for velocity components are valid for all values of the frequencies.

1. Introduction

Exact solutions for the flow due to a single disk in a variety of situations have been obtained by a number of researchers. Berker (1963) has considered the flow due to non-coaxial rotations of a disk and a fluid at infinity and implied the possibility of an exact solution to the Navier-Stokes equations. Thornley (1968) has studied the flow due to non-torsional oscillations of a single disk in a semi-infinite expanse of fluid in a rotating frame of reference. Murthy and Ram (1978) have considered the magnetohydrodynamic flow and heat transfer due to eccentric rotations of a porous disk and a fluid at infinity. Rajagopal (1982) has considered the flow of a simple fluid in an orthogonal rheometer. The flows of Newtonian and non-Newtonian fluids between parallel disks rotating about a common axis have been reviewed by Rajagopal (1992). MHD flow of non-Newtonian fluids was probably first considered by Sarpkaya (1961). The unsteady flow due to non-coaxial rotations of a disk, and a fluid at infinity has been investigated by Kasiviswanathan and Rao (1987), Pop (1979) and Erdogan (1995; 1997).

In the present paper, an analytical solution to the time-dependent equations is given for the third grade flow due to non-coaxial rotations of a porous disk, executing oscillations in its own plane, and a fluid at infinity. An analytical solution is obtained by the perturbation method. The work presented in this paper is a generalization of the work performed by Kasiviswanathan and Rao. To our knowledge, the problem of Kasiviswanathan and Rao (1987) is not yet attempted even for the case of a second grade fluid. The second generalization is concerned to discuss the influence of externally applied magnetic fieldonthe velocity distribution. It is found that asymptotic resonant solution for blowing is possible.

2. Problem formulation

We consider the flow due to an oscillating porous disk lying in the OXY plane rotating about the OZ axis perpendicular to the disk with angular velocity Ω in the Cartesian coordinate system. The fluid at $z=\infty$ rotates about an axis parallel to OZ passing through the point (x_1, y_1). The fluid is electrically conducting and assumed to be permeated by a magnetic field $\boldsymbol{B_0}$ having no components in the x and y directions. For this motion, the velocity field $\boldsymbol{V}$ has the form (Erdogan, 1995; 1997)

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$$u = -\Omega y + f(z,t), \qquad v = \Omega x + g(z,t), \qquad w = -W_0 \tag{2.1}$$

where *u*, *v* and *w* are the velocity components in the *x*, *y* and *z*-directions, and $W_0 > 0$ indicates suction at the disk and $W_0 < 0$ indicates blowing.

The governing equations of motion and Maxwell's equations are

$$\mathrm{div}\boldsymbol{V} = 0, \tag{2.2}$$

$$\rho \frac{d\boldsymbol{V}}{dt} = \mathrm{div}\boldsymbol{S} + \boldsymbol{j} \times \boldsymbol{B}, \tag{2.3}$$

$$\mathrm{div}\boldsymbol{B} = 0, \qquad \mathrm{curl}\boldsymbol{B} = \mu_1 j, \qquad \mathrm{curl}\boldsymbol{E} = -\frac{\partial \boldsymbol{B}}{\partial t}, \tag{2.4}$$

$$\boldsymbol{j} = \sigma(\boldsymbol{E} + \boldsymbol{V} \times \boldsymbol{B}) \tag{2.5}$$

where μ_1 is the magnetic permeability, $\boldsymbol{E}$ is the electric field, σ is the electric conductivity, $\boldsymbol{j}$ is the electric current density, $\boldsymbol{B}$ is the total magnetic field so that $\boldsymbol{B} = \boldsymbol{B}_0 + \boldsymbol{b}$, $\boldsymbol{B}_0$ and $\boldsymbol{b}$ are the imposed and induced magnetic fields respectively.

In our analysis we assume that the fluid is thermodynamically compatible; hence the stress constitutive relation is related in the following manner (Fosdick and Rajagopal, 1980; Dunn and Rajagopal, 1995)

$$\boldsymbol{S} = -p\boldsymbol{I} + \boldsymbol{T}, \qquad \boldsymbol{T} = \mu \boldsymbol{A}_1 + \alpha_1 \boldsymbol{A}_2 + \alpha_2 \boldsymbol{A}_1^2 + \beta_3 \left(tr\boldsymbol{A}_1^2\right)\boldsymbol{A}_1 \tag{2.6}$$

where p is the pressure, $\boldsymbol{I}$ the identity tensor, μ the coefficient of shear viscosity; and α_1, α_2 and β_3 are the material constants which satisfy

$$\mu \ge 0, \qquad \alpha_1 \ge 0, \qquad \beta_3 \ge 0, \qquad -\sqrt{24\mu\beta_3} \le \alpha_1 + \alpha_2 \le \sqrt{24\mu\beta_3}, \tag{2.7}$$

and the specific Helmholtz free energy Ψ has the form

$$\Psi = \hat{\Psi}(\theta, \boldsymbol{L}) + \frac{\alpha_1}{4\rho}\left(\boldsymbol{L} + \boldsymbol{L}^T\right)^2, \qquad \boldsymbol{L} = \mathrm{grad}\boldsymbol{V}. \tag{2.8}$$

The Rivlin-Ericksen tensors (A_n) are defined bythe recursion relation

$$\begin{aligned} \boldsymbol{A}_n &= \frac{d\boldsymbol{A}_{n-1}}{dt} + \boldsymbol{A}_{n-1}(\mathrm{grad}\boldsymbol{V}) + (\mathrm{grad}\boldsymbol{V})^T \boldsymbol{A}_{n-1}, \qquad n > 1, \\ \boldsymbol{A}_1 &= (\mathrm{grad}\boldsymbol{V}) + (\mathrm{grad}\boldsymbol{V})^T. \end{aligned} \tag{2.9}$$

Making use of Eq.(2.1) the equation of continuity (2.2) is satisfied identically and substituting Eqs (2.1), and (2.6) into Eq.(2.3) we obtain

$$\frac{\partial^2 F}{\partial z \partial t} - W_0 \frac{\partial^2 F}{\partial z^2} + i\Omega \frac{\partial F}{\partial z} = \nu \frac{\partial^3 F}{\partial z^3} + \frac{\alpha_1}{\rho}\left(\frac{\partial^4 F}{\partial z^3 \partial t} - W_0 \frac{\partial^4 F}{\partial z^4} - i\Omega \frac{\partial^3 F}{\partial z^3}\right) + \\ - \frac{\sigma B_0^2}{\rho}\frac{\partial F}{\partial z} + 2\beta_3 \frac{\partial^2}{\partial z^2}\left(\left(\frac{\partial F}{\partial z}\right)^2 \frac{\partial \bar{F}}{\partial z}\right) \tag{2.10}$$

where

$$F = f + ig, \qquad \bar{F} = f - ig, \tag{2.11}$$

and ν is the kinematic viscosity. For the problem under consideration the boundary conditions are (Kasiviswanathan and Rao, 1987)

$$F = U_0\left(ae^{i\omega t} + be^{-i\omega t}\right) \qquad \text{at} \qquad z = 0, \tag{2.12}$$

$$F = U_0\left(x_1 + iy_1\right) \qquad \text{at} \qquad z = \infty \tag{2.13}$$

where a, b are complex constants, U_0 is the constant velocity and ω is the frequency of the imposed oscillations.

It is possible to obtain an explicit solution for the system of Eqs (2.10) to (2.13). In what follows, we find such an explicit solution. While the results given may seem cumbersome and unwieldy, it is important to bear in mind that the solutions are exact. Of course, it would be possible to solve the partial differential equations numerically, but we would then have to ensure that such a procedure is stable, convergent, etc. In any event, an exact solution is useful to have and can serve as a check of numerical schemes. The solution of the problem consisting of Eqs (2.10) to (2.13) has been obtained employing the procedure used by the authors in reference (Hayat *et al.*, 2001). After lengthy calculations, the velocity field for $\delta > \Omega_1$ are given by

$$u = U_0\left[x_1 - Z_1 e^{-a_0\eta} + Z_2 e^{-a_1\eta} + Z_3 e^{-a_2\eta}\right] + \\ + \in U_0\left\{Z_4 e^{-3a_0\eta} + Z_5 e^{-c_1\eta} + Z_6 e^{-c_2\eta} + Z_7 e^{-c_3\eta} + Z_8 e^{-c_4\eta} + \right. \\ \left. + Z_9 e^{-3a_1\eta} + Z_{10} e^{-c_5\eta} + Z_{11} e^{-c_6\eta} + Z_{12} e^{-3a_2\eta} + Z_{13} e^{-c_7\eta}\right\}, \tag{2.14}$$

$$v = U_0\left[y_1 + Z_{14} e^{-a_0\eta} + Z_{15} e^{-a_1\eta} + Z_{16} e^{-a_2\eta}\right] + \\ + \in U_0\left\{Z_{17} e^{-3a_0\eta} + Z_{18} e^{-c_1\eta} + Z_{19} e^{-c_2\eta} + Z_{20} e^{-c_3\eta} + Z_{21} e^{-c_4\eta} + \right. \\ \left. + Z_{22} e^{-3a_1\eta} + Z_{23} e^{-c_5\eta} + Z_{24} e^{-c_6\eta} + Z_{25} e^{-3a_2\eta} + Z_{26} e^{-c_7\eta}\right\}, \tag{2.15}$$

$$\eta = \frac{zW_0}{\nu}, \qquad \tau = \frac{W_0^2 t}{\nu}, \qquad \delta = \frac{\omega\nu}{W_0^2}, \qquad \alpha = \frac{\alpha_1 W_0^2}{\rho\nu^2},$$

$$\Omega_1 = \frac{\Omega\nu}{W_0^2}, \qquad n = \frac{\sigma B_0^2 \nu}{\rho W_0^2}, \qquad M = \frac{U_0^2 W_0^2}{\rho\nu^3},$$

and the expressions for the constants a_0, a_1 etc. can be routinely determined.

Similarly, the asymptotic solution for $\delta < \Omega_1$, and $\delta = \Omega_1$ exist but is not given explicitly in order to avoid the large size of calculations. Further, the asymptotic solutions for blowing can be obtained in the three cases by replacing W_0 with $-W_0$.

3. Conclusions

We have solved the canonical problem for a time dependent magnetohydrodynamic flow due to non-coaxial rotations of a porous oscillating disk and a fluid at infinity. The velocity field is governed by a fourth order non-linear partial differential equation. The solution to the governing equation for suction/blowing consists of two parts. The first part is the solution for a second grade fluid and second part arises due to a third grade fluid. The most important feature in the case of blowing solution is that the magnetohydrodynamic solution for the resonant case, satisfy the boundary condition at infinity for all values of frequency including the resonant frequency. Consequently, the associated boundary layers remain bounded for the resonant case. In contrast to the hydrodynamic solution for the case of blowing and resonance where the blowing promotes the spreading of the oscillations far away from the disk, the oscillatory boundary layer flows are confined to the ultimate boundary layers for all frequencies including the resonant frequencies. The physical implication of this conclusion is that for the case of resonance, the unbounded spreading of the oscillations away from the disk is controlled by the external magnetic field. The asymptotic analysis in the resonant case further indicates the existence of diffusive hydromagnetic waves. Eventually, these waves are found to decay within the boundary layers. It is of interest to note that the external magnetic field expedites the decay process of these waves.

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