DISPLACEMENT POTENTIAL SOLUTION OF SHORT STIFFENED FLAT COMPOSITE BARS UNDER AXIAL LOADING

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The present paper describes a new approach to analytical solution of two-dimensional stress problems of orthotropic composite materials. In this approach, the elastic problem is formulated in terms of a single potential function, defined in terms of the displacement components, which satisfies a single differential equation of equilibrium. The new mathematical model, namely, the displacement potential formulation is especially suitable for the solution of mixed-boundary-value elastic problems of orthotropic composite materials. This paper presents the solution to stresses and displacements at different sections of short stiffened flat composite bars under axial loading, where a number of bar aspect ratios are considered together with different materials of interest. The solutions are obtained in the form of infinite series and the results are presented mainly in the form of graphs. The results appear to be quite reasonable and accurate, and thus establish the soundness as well as reliability of the present displacement potential approach.

Key words: analytical solution, stress analysis, stiffened flat-bar, displacement potential, orthotropic composite material.

1. Introduction

The use of stiffeners in the construction of engineering structures is quite extensive. In the solution of stiffened structures, the physical conditions of stiffeners are mathematically modeled usually in terms of a mixed mode of boundary conditions, that is, the known normal stress and tangential displacement. However, the earlier mathematical models of elasticity were very deficient in handling the practical stress problems, as most of them are of the mixed-boundary-value type. The mixed-boundary-value problems are those in which the boundary conditions are specified as a mixture of boundary restraints and boundary loading, where the combination of the boundary conditions may also change from segment to segment of the boundary. Since the exact analytical solution of mixed-boundary-value problems, specially with non-isotropic materials is beyond the scope of the existing mathematical models of elasticity, the use of a new mathematical formulation is investigated here in an attempt to analyze the state of stresses as well as deformations in short stiffened flat-bars of composite material under axial loading.

Stress analysis has now become a classical subject in the field of elasticity. But somehow these stress analysis problems are still suffering from a lot of shortcomings and thus are being constantly looked into (Murty, 1984; Suzuki, 1986; Hardy and Pipelzadeh, 1991). Although elasticity problems were formulated long before, exact solutions to practical problems are hardly available because of the inability of managing the associated physical conditions in a justifiable manner. Actually management of boundary conditions is one of the major obstacles to the reliable solution to practical problems. The famous Saint Venant's principle is still applied and its merit is evaluated in solving problems of solid mechanics (Horgan and Knowels, 1983), in which full boundary effects could not be taken into account satisfactorily in the process of solution. Even now, photoelastic studies are being carried out for classical problems like uniformly loaded beams on two supports (Durelli and Ranganayakamma, 1987; 1989) mainly because the boundary effects could not be taken into account fully in their analytical method of solutions.

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Elasticity problems are usually formulated either in terms of deformation parameters or stress parameters. Among the existing mathematical models of plane boundary-value stress problems, the stress function approach (Timoshenko and Goodier, 1979) and the displacement formulation (Uddin, 1966) are noticeable. The shortcoming of the stress function approach is that it accepts boundary conditions only in terms of loadings. Boundary restraints specified in terms of the displacement components cannot be satisfactorily imposed on the stress function. As most of the practical problems of elasticity are of mixed boundary conditions, the approach fails to provide any explicit understanding of the state of stresses at the critical regions of supports and loadings. The displacement formulation, on the other hand, involves finding two displacement functions simultaneously from the two second-order elliptic partial differential equations of equilibrium, which is extremely difficult, and this problem becomes more serious when the boundary conditions are mixed (Uddin, 1966). The difficulties involved in trying to solve practical stress problems using the existing models are clearly pointed out by Durelli and Ranganayakamma (1989) and also by us in our previous reports (Ahmed, 1993; Idris, 1993; Ahmed *et al*. 1996a; Idris *et al*., 1996).

As stated above, neither of the formulations is suitable for solving problems of mixed-boundary conditions, and hence a new mathematical model is used to solve the present problem of composite structure. In this approach, the plane elastic problem is formulated in terms of a potential function of space variables, defined in terms of the two displacement components. It should be mentioned that the present modeling approach enables us to manage the mixed mode of the boundary conditions as well as their zones of transition very efficiently. The present paper demonstrates the application of the model for the analytical solution of short flat-bars of orthotropic composite material, subjected to axial loading. The supporting edge of the bar is assumed to be rigidly fixed and the two opposing edges are stiffened. The solutions are obtained in the form of infinite series and the corresponding distributions of different stress and displacement components are presented mainly in the form of graphs. Solutions of different parameters of interest are obtained for a different aspect ratio of the bar and also for different materials of interest. In an attempt to demonstrate the effect of material orthotropy, solutions of different composite materials are compared with that for a corresponding isotropic material. It is worth mentioning that the recent research and developments in using the displacement potential approach have generated much renewed interest in the field of both analytical and numerical solutions of practical stress problems (Idris *et al*., 1996; Ahmed *et al*., 1996a; 1996b; 1998; 1999; 2005; Akanda *et al*., 2000; 2002). The present paper is an attempt to extend the capability of our displacement potential formulation to include the problems of orthotropic composite materials.

2. Displacement potential formulation for orthotropic materials

The stress at a point in a two-dimensional body, developed due to its interaction with the external forces and restraints on its boundaries, is represented by three dependent variables, namely, σ_{xx}, σ_{yy} and σ_{xy}, as shown in Fig.1. With reference to a rectangular coordinate system, in the absence of body forces, these three variables are governed by the following two equilibrium and one compatibility equations (Timoshenko and Goodier, 1979)

$$\frac{\partial\sigma_{xx}}{\partial x}+\frac{\partial\sigma_{xy}}{\partial y}=0\,, \tag{2.1a}$$

$$\frac{\partial\sigma_{yy}}{\partial y}+\frac{\partial\sigma_{xy}}{\partial x}=0\,, \tag{2.1b}$$

$$\left(\frac{\partial^2}{\partial x^2}+\frac{\partial^2}{\partial y^2}\right)\left(\sigma_{xx}+\sigma_{yy}\right)=0\,. \tag{2.1c}$$

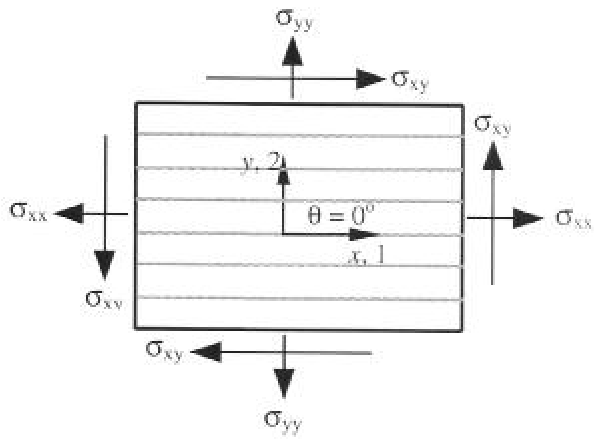


Fig.1. Stress components on a plane of an orthotropic composite material.

Instead of solving for three parameters simultaneously from the above three differential equations, the practical approach is to transform the three equations into two, containing two displacement parameters, u_x and u_y, as the unknowns. In that case, only the first two equilibrium conditions of Eqs (2.1) are relevant in obtaining the two displacement parameters, as the remaining Eq.(2.1c) establishes only their continuity in the case of stress formulation and thus is irrelevant here.

To express the equilibrium equations in terms of displacement components, we need to express the three stress components in terms of displacement parameters. The corresponding three stress-displacement relations for general orthotropic materials are obtained from Hooke's law as follows (Jones, 1975)

$$\sigma_{xx} = \frac{E_{11}}{1-\mu_{12}\,\mu_{21}}\left[\frac{\partial u_x}{\partial x} + \mu_{21}\frac{\partial u_y}{\partial y}\right], \tag{2.2a}$$

$$\sigma_{yy} = \frac{E_{22}}{1-\mu_{12}\,\mu_{21}}\left[\frac{\partial u_y}{\partial y} + \mu_{12}\frac{\partial u_x}{\partial x}\right], \tag{2.2b}$$

$$\sigma_{xy} = G_{12}\left[\frac{\partial u_x}{\partial y} + \frac{\partial u_y}{\partial x}\right]. \tag{2.2c}$$

Substituting the above stress-displacement relations into Eqs (2.1a) and (2.1b), and using the reciprocal relation, $E_{22}\mu_{12} = E_{11}\,\mu_{21}$, we obtain the two equilibrium equations for two-dimensional problems of orthotropic materials in terms of the two displacement components as follows (Timoshenko and Goodier, 1979; Jones, 1975)

$$\left(\frac{E_{11}^2}{E_{11}-\mu_{12}^2 E_{22}}\right)\frac{\partial^2 u_x}{\partial x^2} + \left(\frac{\mu_{12}\,E_{11}E_{22}}{E_{11}-\mu_{12}^2 E_{22}} + G_{12}\right)\frac{\partial^2 u_y}{\partial x \partial y} + G_{12}\frac{\partial^2 u_x}{\partial y^2} = 0\,, \tag{2.3a}$$

$$\left(\frac{E_{11}E_{22}}{E_{11}-\mu_{12}^2 E_{22}}\right)\frac{\partial^2 u_y}{\partial y^2}+\left(\frac{\mu_{12}E_{11}E_{22}}{E_{11}-\mu_{12}^2 E_{22}}+G_{12}\right)\frac{\partial^2 u_x}{\partial x\partial y}+G_{12}\frac{\partial^2 u_y}{\partial x^2}=0\,. \tag{2.3b}$$

Although the above two differential equations are theoretically sufficient to solve the mixed-boundary-value elastic problems of orthotropic composite materials, but, in reality, it is extremely difficult to solve for two functions simultaneously satisfying the two second-order elliptic partial differential equations. In order to overcome this difficulty, the existence of a new potential function of space variables is investigated in an attempt to reduce the problem to the determination of a single variable from a single differential equation of equilibrium. In the present formulation, a new potential function $\psi(x, y)$ is thus defined in terms of the two displacement components following the procedure of Ahmed *et al.* (1998) as follows

$$u_x=\frac{\partial^2\psi}{\partial x\partial y}, \tag{2.4a}$$

$$u_y=-\frac{1}{Z_{11}}\left[E_{11}^2\frac{\partial^2\psi}{\partial x^2}+G_{12}\left(E_{11}-\mu_{12}^2E_{22}\right)\frac{\partial^2\psi}{\partial y^2}\right] \tag{2.4b}$$

where, $Z_{11}=\mu_{12}E_{11}E_{22}+G_{12}\left(E_{11}-\mu_{12}^2E_{22}\right)$.

With the above definition of $\psi(x, y)$, the first equilibrium Eq.(2.3a) is automatically satisfied. Therefore, ψ has to satisfy the second equilibrium Eq.(2.3b) only. Expressing Eq.(2.3b) in terms of the potential function ψ, the condition that ψ has to satisfy becomes

$$E_{11}G_{12}\frac{\partial^4\psi}{\partial x^4}+E_{22}\left(E_{11}-2\mu_{12}G_{12}\right)\frac{\partial^4\psi}{\partial x^2\partial y^2}+E_{22}G_{12}\frac{\partial^4\psi}{\partial y^4}=0\,. \tag{2.5}$$

Therefore, the problem is thus reduced to the evaluation of a single variable $\psi(x, y)$ from a single fourth-order partial differential equation of equilibrium Eq.(2.5). The corresponding governing differential equation for the isotropic elastic solids can readily be obtained when the elastic constants in the two directions are assumed to be identical $\left(E_{11}=E_{22}=E;\ \mu_{12}=\mu_{21}=\mu;\ G_{12}=G=E/[2(1+\mu)]\right)$, which is as follows

$$\frac{\partial^4\psi}{\partial x^4}+2\frac{\partial^4\psi}{\partial x^2\partial y^2}+\frac{\partial^4\psi}{\partial y^4}=0\,. \tag{2.6}$$

It is noted that the above bi-harmonic partial differential equation has been used extensively in our displacement potential formulation for the solution of mixed-boundary-value elastic problems of isotropic solids. The reliability as well as the suitability of the formulation has been verified repeatedly in our previous research (Ahmed *et al.*, 1996a; 1996b; 1998; 1999; Idris *et al.*, 1996; Akanda *et al.*, 2000; 2002).

3. Boundary conditions in terms of the displacement potential

The physical conditions that exist on the boundaries of an elastic body are usually visualized in two different ways, namely, (a) the boundary restraints and (b) boundary loading, that is, known displacements and stresses on the boundary, respectively. Both the displacements and stresses are usually defined by their normal and tangential components.

The solution of the governing differential equation (5) requires specification of two independent conditions at each boundary segment of the elastic body. Four components, taken two at a time, will create six different boundary conditions. Out of these six boundary conditions, the possible four boundary conditions that may occur in practice are: (a) normal displacement and tangential displacement, (b) normal displacement and tangential stress, (c) tangential displacement and normal stress, and (d) normal stress and tangential stress

Since our objective is to solve the problem in terms of the displacement potential, all the components of interest are required to be expressed in terms of the function, ψ. With reference to a rectangular coordinate system, the explicit expressions for the displacement components, u_x and u_y, in terms of the function, $\psi(x, y)$ are as follows

$$u_x(x, y) = \frac{\partial^2 \psi}{\partial x \partial y}, \tag{3.1a}$$

$$u_y(x, y) = -\frac{1}{Z_{11}}\left[E_{11}^2 \frac{\partial^2 \psi}{\partial x^2} + G_{12}\left(E_{11} - \mu_{12}^2 E_{22}\right)\frac{\partial^2 \psi}{\partial y^2}\right]. \tag{3.1b}$$

Combining Eqs (2.2) and (2.4), the explicit expressions of the three stress components in terms of the potential function, ψ are obtained as follows

$$\sigma_{xx}(x, y) = \frac{E_{11}G_{12}}{Z_{11}}\left[E_{11}\frac{\partial^3 \psi}{\partial x^2 \partial y} - \mu_{12}E_{22}\frac{\partial^3 \psi}{\partial y^3}\right], \tag{3.2a}$$

$$\sigma_{yy}(x, y) = \frac{E_{11}E_{22}}{Z_{11}}\left[\left(\mu_{12}G_{12} - E_{11}\right)\frac{\partial^3 \psi}{\partial x^2 \partial y} - G_{12}\frac{\partial^3 \psi}{\partial y^3}\right], \tag{3.2b}$$

$$\sigma_{xy}(x, y) = -\frac{E_{11}G_{12}}{Z_{11}}\left[E_{11}\frac{\partial^3 \psi}{\partial x^3} - \mu_{12}E_{22}\frac{\partial^3 \psi}{\partial x \partial y^2}\right]. \tag{3.2c}$$

The corresponding expressions for the displacement and stress boundary conditions for the isotropic materials can be obtained when the respective conditions are substituted in the above expressions Eqs (3.1)-(3.2), which are as follows

$$u_x(x, y) = \frac{\partial^2 \psi}{\partial x \partial y}, \tag{3.3a}$$

$$u_y(x, y) = -\frac{1}{1+\mu}\left[2\frac{\partial^2 \psi}{\partial x^2} + (1-\mu)\frac{\partial^2 \psi}{\partial y^2}\right], \tag{3.3b}$$

$$\sigma_{xx}(x, y) = \frac{E}{(1+\mu)^2}\left[\frac{\partial^3 \psi}{\partial x^2 \partial y} - \mu\frac{\partial^3 \psi}{\partial y^3}\right], \tag{3.3c}$$

$$\sigma_{yy}(x,\, y) = -\frac{E}{(1+\mu)^2}\left[(2+\mu)\frac{\partial^3\psi}{\partial x^2 \partial y} + \frac{\partial^3\psi}{\partial y^3}\right], \tag{3.3d}$$

$$\sigma_{xy}(x,\, y) = -\frac{E}{(1+\mu)^2}\left[\frac{\partial^3\psi}{\partial x^3} - \mu\frac{\partial^3\psi}{\partial x \partial y^2}\right]. \tag{3.3e}$$

From the above expressions of boundary conditions, it is revealed that there is no technical difficulty in satisfying all the modes of boundary conditions appropriately. Moreover, when compared to the approach of solving the problem in terms of displacement components, it has the advantage that only one function is required to be evaluated instead of solving for two variables simultaneously.

4. Solution of the stiffened composite bar problem

The problem of a rectangular flat-bar of orthotropic composite material is considered here, where the two opposing edges of the bar are stiffened, while the boundary conditions at the other two edges are kept unspecified at the moment. The flat bar is considered to be of unit thickness and its configuration with respect to co-ordinate axes is illustrated in Fig.2.

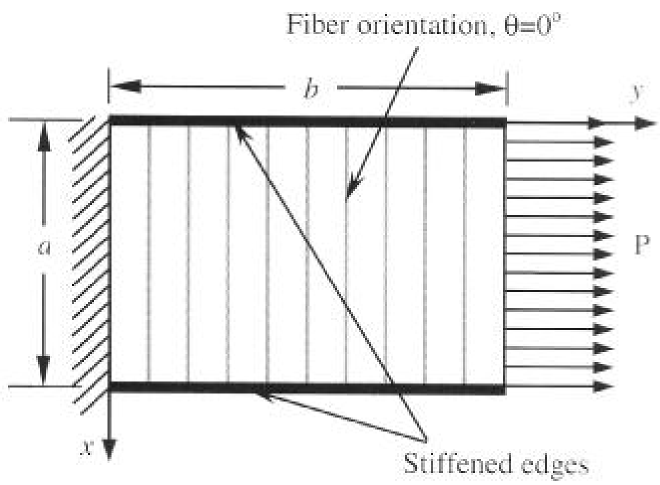


Fig.2. A flat composite bar with two opposing edges stiffened, under uniform axial loading.

In this case, if the potential function, ψ is assumed to be

$$\psi = \sum_{m=1}^{\infty} Y_m \sin\alpha x \tag{4.1}$$

where, Y_m is a function of y only, and $\alpha = m\pi/a$, then Y_m has to satisfy the ordinary differential equation

$$\frac{E_{22}}{E_{11}} Y_m'''' - \left(\frac{E_{22}}{G_{12}} - \frac{2\mu_{12}E_{22}}{E_{11}}\right)\alpha^2 Y_m'' + \alpha^4 Y_m = 0\,. \tag{4.2}$$

The general solution of the differential Eq.(4.2) can be given by

$$Y_m = A_m e^{m_1 y} + B_m e^{m_2 y} + C_m e^{m_3 y} + D_m e^{m_4 y} \tag{4.3}$$

where

$$m_1, m_2 = \alpha\left[\frac{E_{11}}{E_{22}}\left\{K_{11} \pm \sqrt{K_{11}^2 - \frac{4E_{22}}{E_{11}}}\right\}\right]^{1/2},$$

$$m_3, m_4 = -\alpha\left[\frac{E_{11}}{E_{22}}\left\{K_{11} \pm \sqrt{K_{11}^2 - \frac{4E_{22}}{E_{11}}}\right\}\right]^{1/2},$$

$K_{11} = \dfrac{E_{22}}{G_{12}} - \dfrac{2\mu_{12}E_{22}}{E_{11}}$, and A_m, B_m, C_m, D_m are arbitrary constants.

Now combining Eqs (3.1)-(3.2) and (4.1), the expressions of stress and displacement components are obtained as follows

$$u_x(x, y) = \sum_{m=1}^{\infty} \alpha Y_m' \cos\alpha x, \tag{4.4}$$

$$u_y(x, y) = -\frac{1}{Z_{11}} \sum_{m=1}^{\infty} \left[Z_{22} Y_m'' - E_{11}^2 \alpha^2 Y_m\right] \sin\alpha x, \tag{4.5}$$

$$\sigma_{xx}(x, y) = \frac{E_{11}G_{12}}{Z_{11}} \sum_{m=1}^{\infty} \left[\left\{\mu_{12}E_{22}Y_m''' + \alpha^2 E_{11} Y_m'\right\}\sin\alpha x\right], \tag{4.6}$$

$$\sigma_{yy}(x, y) = \frac{E_{11}E_{22}}{Z_{11}} \sum_{m=1}^{\infty} \left[\left\{G_{12}Y_m''' + \alpha^2(\mu_{12}G_{12} - E_{11})Y_m'\right\}\sin\alpha x\right], \tag{4.7}$$

$$\sigma_{xy}(x, y) = \frac{E_{11}G_{12}}{Z_{11}} \sum_{m=1}^{\infty} \left[\left\{\alpha\mu_{12}E_{22}Y_m'' + \alpha^3 E_{11} Y_m'\right\}\cos\alpha x\right] \tag{4.8}$$

where, $Z_{22} = G_{12}\left(E_{11} - \mu_{12}^2 E_{22}\right)$, and the ($'$) indicates differentiation with respect to y.

Substituting the different derivatives of Y_m in the expressions of the stress and displacement components Eqs (4.4)-(4.8), we get

$$u_x(x, y) = \sum_{m=1}^{\infty}\left[\left(m_1 A_m e^{m_1 y} + m_2 B_m e^{m_2 y} + m_3 C_m e^{m_3 y} + m_4 D_m e^{m_4 y}\right)\alpha\cos\alpha x\right], \tag{4.9}$$

$$u_y(x, y) = -\frac{1}{Z_{11}}\left[\sum_{m=1}^{\infty}\left\{\begin{matrix}\left(m_1^2 Z_{22} - \alpha^2 E_{11}^2\right)A_m e^{m_1 y} + \left(m_2^2 Z_{22} - \alpha^2 E_{11}^2\right)B_m e^{m_2 y} + \\ + \left(m_3^2 Z_{22} - \alpha^2 E_{11}^2\right)C_m e^{m_3 y} + \left(m_4^2 Z_{22} - \alpha^2 E_{11}^2\right)D_m e^{m_4 y}\end{matrix}\right\}\sin\alpha x\right], \tag{4.10}$$

$$\sigma_{xx}(x, y) = \frac{E_{11}G_{12}}{Z_{11}}\left[\sum_{m=1}^{\infty}\left\{\begin{matrix}\left(m_1\alpha^2 E_{11} + m_1^3\mu_{12}E_{22}\right)A_m e^{m_1 y} + \\ + \left(m_2\alpha^2 E_{11} + m_2^3\mu_{12}E_{22}\right)B_m e^{m_2 y} + \\ + \left(m_3\alpha^2 E_{11} + m_3^3\mu_{12}E_{22}\right)C_m e^{m_3 y} + \\ + \left(m_4\alpha^2 E_{11} + m_4^3\mu_{12}E_{22}\right)D_m e^{m_4 y}\end{matrix}\right\}\sin\alpha x\right], \tag{4.11}$$

$$\sigma_{yy}(x, y) = \frac{E_{11}E_{22}}{Z_{11}}\left[\sum_{m=1}^{\infty}\left\{\begin{matrix}\left(m_1\alpha^2(\mu_{12}G_{12} - E_{11}) + m_1^3 G_{12}\right)A_m e^{m_1 y} + \\ + \left(m_2\alpha^2(\mu_{12}G_{12} - E_{11}) + m_2^3 G_{12}\right)B_m e^{m_2 y} + \\ + \left(m_3\alpha^2(\mu_{12}G_{12} - E_{11}) + m_3^3 G_{12}\right)C_m e^{m_3 y} + \\ + \left(m_4\alpha^2(\mu_{12}G_{12} - E_{11}) + m_4^3 G_{12}\right)D_m e^{m_4 y}\end{matrix}\right\}\sin\alpha x\right], \tag{4.12}$$

$$\sigma_{xy}(x, y) = -\frac{E_{11}G_{12}}{Z_{11}}\left[\sum_{m=1}^{\infty}\left\{\begin{matrix}\left(\alpha^3 E_{11} + \alpha m_1^2\mu_{12}E_{22}\right)A_m e^{m_1 y} + \\ + \left(\alpha^3 E_{11} + \alpha m_2^2\mu_{12}E_{22}\right)B_m e^{m_2 y} + \\ + \left(\alpha^3 E_{11} + \alpha m_3^2\mu_{12}E_{22}\right)C_m e^{m_3 y} + \\ + \left(\alpha^3 E_{11} + \alpha m_4^2\mu_{12}E_{22}\right)D_m e^{m_4 y}\end{matrix}\right\}\cos\alpha x\right]. \tag{4.13}$$

For the present problem, it is seen that the boundary conditions on the two stiffened edges

$$u_x(x, y) = \sigma_{xx}(x, y) = 0; \quad \text{at} \quad x = 0 \quad \text{and} \quad x = a,$$

are satisfied automatically. We are now in a position to apply any feasible boundary conditions on the left and right lateral boundaries of the bar, i.e., at $y = 0$ and $y = b$, and thereby determine the values of the constants A_m, B_m, C_m, and D_m. For the stiffened bar problem, where the left supporting boundary, $y = 0$, $y = b$ is rigidly fixed, as shown in Fig.2, the associated boundary conditions are

$$u_x(x, 0) = u_y(x, 0) = 0.$$

Now, the axial loading on the right lateral boundary of the bar, $y = b$ can be expressed mathematically as follows

$$\begin{aligned}\sigma_{yy}(x, b) &= f(x) = E_0 + \sum_{m=1}^{\infty} E_m \sin\alpha x = P, \\ \sigma_{xy}(x, b) &= 0,\end{aligned} \tag{4.14}$$

which represents all possible normal loading on the boundary. In our present problem, $E_0 = 0$, and

$$E_m = \frac{2}{a}\int_0^a f(x)\sin\alpha x\,dx = \frac{4P}{m\pi}. \tag{4.15}$$

Substituting the above conditions of the left and right lateral edges into the general expressions of Eqs (4.9), (4.10), (4.12), and (4.13), we get the following four simultaneous algebraic equations in terms of the four unknown coefficients

$$\begin{bmatrix} m_1 & m_2 & m_3 & m_4 \\ P_1 & P_2 & P_3 & P_4 \\ Q_1 & Q_2 & Q_3 & Q_4 \\ R_1 & R_2 & R_3 & R_4 \end{bmatrix} \begin{Bmatrix} A_m \\ B_m \\ C_m \\ D_m \end{Bmatrix} = \begin{Bmatrix} 0 \\ 0 \\ \bar{E}_m \\ 0 \end{Bmatrix} \tag{4.16}$$

where

$$\left.\begin{aligned} P_i &= m_i^2 Z_{22} - \alpha^2 E_{11}^2 \\ Q_i &= \left\{m_i\alpha^2\left(\mu_{12}G_{12} - E_{11}\right) + m_i^3 G_{12}\right\}e^{m_i b} \\ R_i &= \left(\alpha^3 E_{11} + \alpha m_i^2 \mu_{12} E_{22}\right)e^{m_i b} \\ \bar{E}_m &= E_m Z_{11}/E_{11}E_{22} \end{aligned}\right] \quad i = 1, 2, 3, 4.$$

Solution of the above algebraic Eq.(4.16) yields the unknown constants in the evaluation, A_m, B_m, C_m, and D_m. Once the values of the unknowns are known, the explicit expressions of the different parameters of interest are readily obtained; which are valid for the entire region of the stiffened composite bar. The solutions of the problem are thus obtained as a function of the elastic properties of the composite material, bar aspect ratio, space variables (x, y), and the loading parameter.

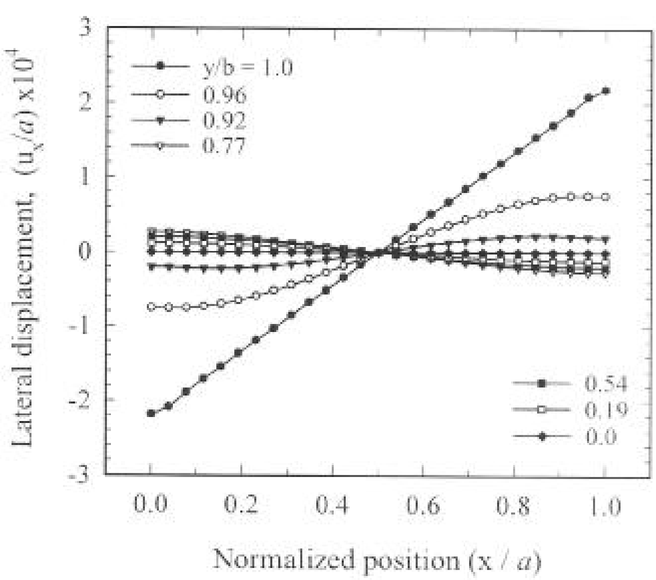
5. Results and discussions

The solution of the stiffened composite bar problem as obtained by the present analytical method of solution is presented in this section. In order to present the solutions graphically, the analytical solutions obtained are evaluated numerically. In order to make the results non-dimensional, the displacements are expressed as the ratio of actual displacement to the actual dimension of the bar (a), and the stresses are expressed as the ratio of the actual stress to the applied loading parameter, P. The value of the loading parameter is taken to be 6000 psi and it is kept constant for the whole analysis. The different elastic properties of different composite materials used in the present analysis are listed in Tab.1.

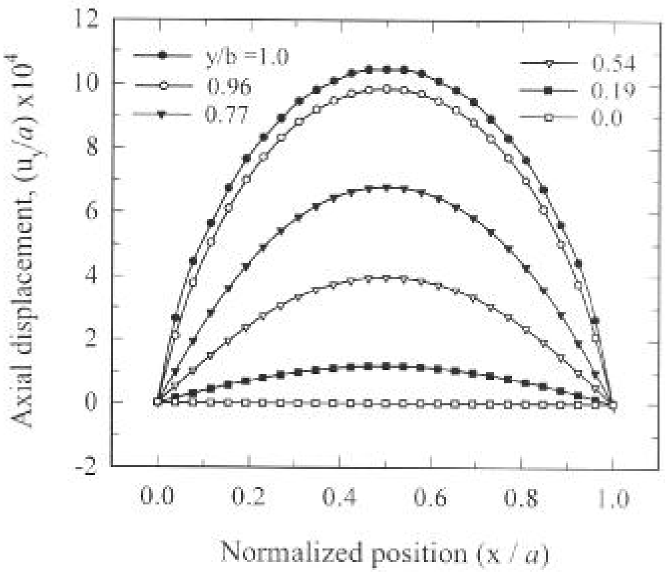
Both the distribution of the normalized lateral and axial displacement components are observed to be in good agreement with the physical model of the problem, as seen in Figs 3a and b, respectively. At the mid-section of the bar $(x/a = 0.5)$, the lateral displacement is always zero and it is maximum at the two opposite stiffened edges, which makes the distribution antisymmetric about the mid-section (Fig.3a). For sections, $0.9 \le y/b \le 1.0$, the lateral displacement is found to be negative for the upper half but positive for the lower half of the bar, and its magnitude decreases with decreasing the value of y. However, for section, $y/b < 0.9$, the displacement is positive for the upper and negative for the lower half, completely reverse to that found for a small region just immediately left to the loaded boundary. Tensile loading in axial direction should have normally led to contraction in the x-direction due to the effect of Poisson's ratio. This expectation is found to be true over the range $0 < y/b < 0.9$. But, for the small region, $y/b \geq 0.9$, the bar is observed to be expanding in the x-direction, which is in contrast to our general intuition and may be attributed to the physical conditions of the stiffened boundary under tension. As appears from Fig.3b, the axial displacements at the upper and lower stiffened boundaries are completely zero, which reflects the effect of stiffeners appropriately in the solution. Although the uniformly distributed axial loading is applied on the right lateral edge, the stiffeners at the two opposing boundaries make the distribution parabolic having maximum magnitude at the middle and zero at the two ends. The solutions for both the displacement components are found to be zero at the fixed support, which is also in conformity with the physical characteristic of the problem.

Table 1. Properties of composites used to obtain numerical results.

Material	Property	Composite		
		Glass-Epoxy	Boron-Epoxy	Graphite-Epoxy
Fiber	$E_f\left(10^6\ psi\right)$	*12.4*	*60.0*	*140.0*
	μ_f	*0.22*	*0.20*	*0.20*
Matrix	$E_m\left(10^6\ psi\right)$	*0.50*	*0.50*	*0.50*
	μ_m	*0.35*	*0.35*	*0.35*
Composite	$E_{11}\left(10^6\ psi\right)$	*8.6*	*41.0*	*94.0*
	$E_{22}\left(10^6\ psi\right)$	*3.2*	*3.5*	*3.6*
	$E_f\left(10^6\ psi\right)$	*1.3*	*1.5*	*1.6*
	μ_{12}	*0.26*	*0.27*	*0.25*
	μ_{21}	*0.047*	*0.023*	*0.010*



(a) Lateral displacement component.

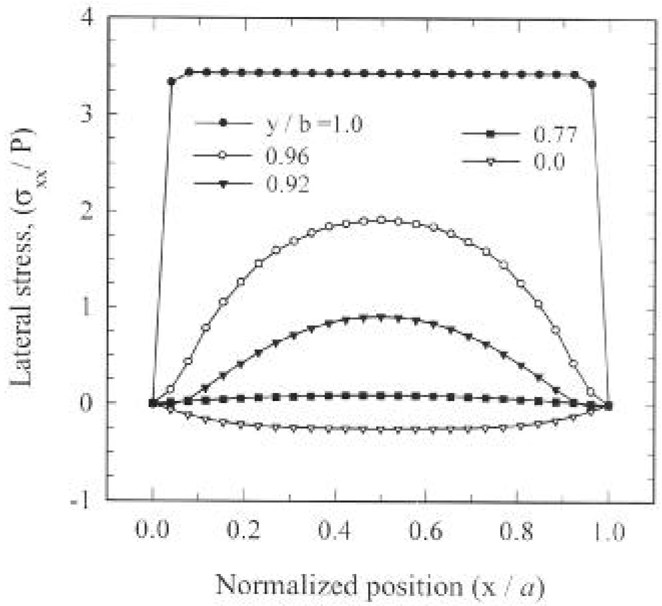


(b) Axial displacement component.

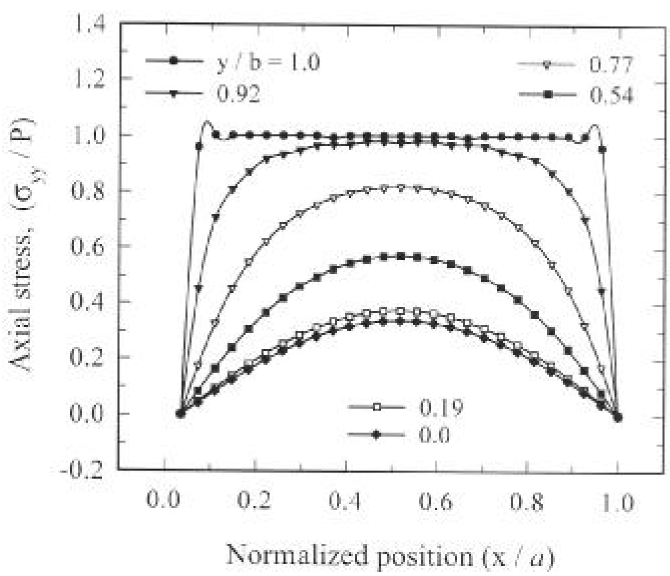
Fig.3. Distribution of normalized displacement components at different sections of the Boron-Epoxy composite bar, $b/a = 1$.

Figure 4 illustrates the distribution of different stress components with respect to *x*, for the Boron-Epoxy composite bar, $b/a = 1$. The overall distribution of the normal stress component, σ_{xx}, as shown in Fig.4a, reveals that the major portion of the composite bar is under compression, as the stress component is negative for sections $0 \le y/b \le 0.7$. However, for the remaining portion of the bar, that is, in the neighborhood of the loaded boundary, $0.7 < y/b \le 1$, the bar is found to be in tension, within which the stress level changes significantly with the change of *y* values. The magnitude of the tensile stress is however found to be much higher than that of compression. The boundary under loading, $y/b = 1$, is identified here to be the most critical section of the stiffened bar in terms of the stress component, where the magnitude of the stress in the lateral direction is more than three times of the applied loading. The magnitude of the stress is however found to decrease as we move towards the supporting zone.

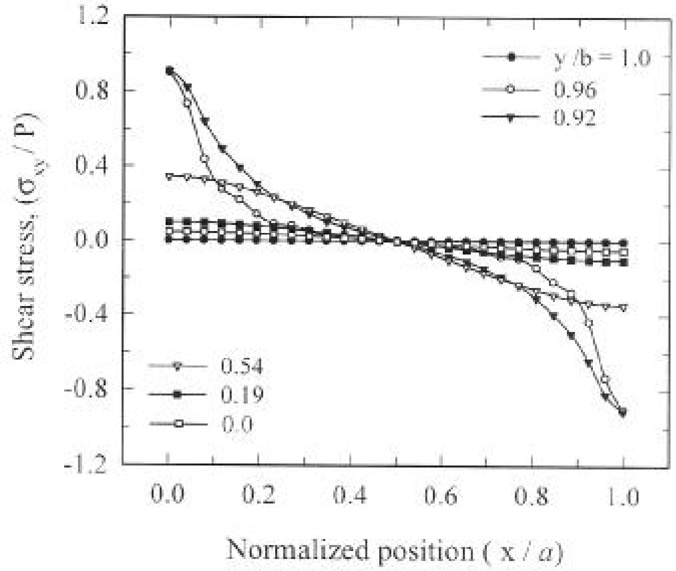
The distribution of the axial stress component (σ_{yy}/P) is also found to be in good agreement with the physical characteristic of the stiffened bar, that is, maximum at the right lateral end and minimum at the left supporting edge (see Fig.4b). The stress at the stiffened edges is completely zero but it is maximum at the mid-section, $x/a = 0.5$, which make the distribution nearly parabolic with respect to *x*-axis. From the distribution of shearing stress component, as shown in Fig.4c, it is seen that its value is zero at the axis of symmetry, $x/a = 0.5$ and maximum at the two opposing stiffened boundaries, which makes the distribution antisymmetric about the axis of symmetry. The shearing stress at the right loaded boundary is found to be completely zero and that at the left supporting end is insignificant, which verifies the solution to be in good conformity with the physical model of the problem.



(a) Lateral stress component.



(b) Axial stress component.

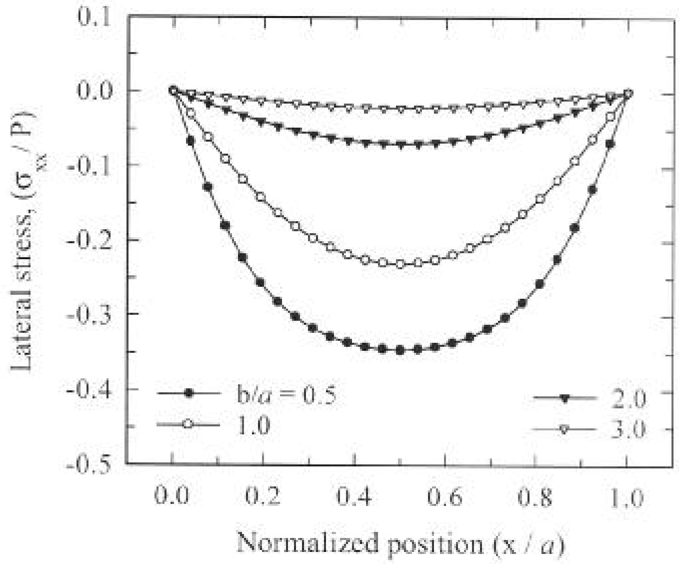


(c) Shear stress component.

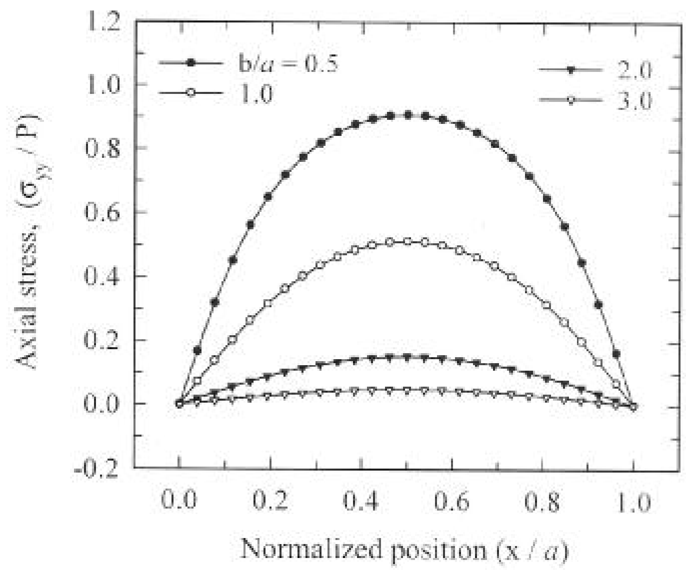
Fig.4. Distribution of normalized stress components at different sections of Boron-Epoxy composite bar, $b/a = 1$.

5.1. Effect of bar aspect ratio on the solution

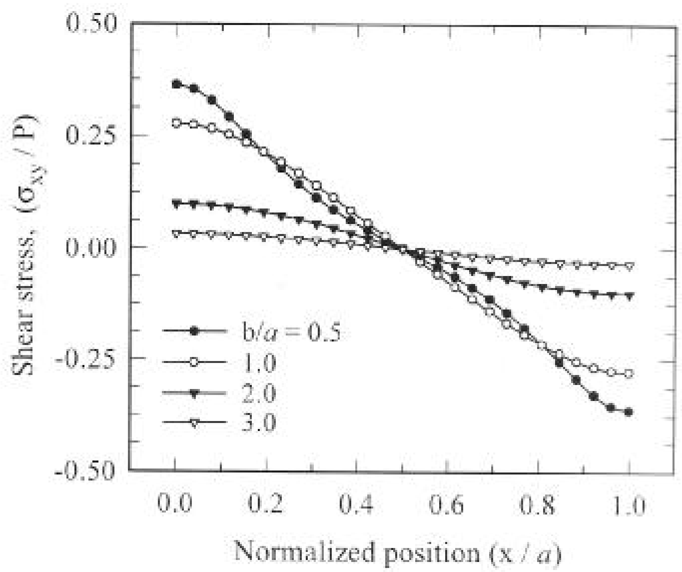
Figure 5 describes the effect of aspect ratio (b/a) of the stiffened composite bar on the solutions of stress components. For the sake of convenience, solutions are presented only for the mid-longitudinal section of the Boron-Epoxy composite bars. For all the bars considered, the mid-sections are found to be under compression as far as the lateral stress is concerned (Fig.5a). However, the magnitude of the stress is maximum for the smallest bar $(b/a = 0.5)$ and minimum for the largest one $(b/a = 3.0)$. As shown in Fig.5b, a similar trend is observed for the axial stress component when the corresponding solutions are analyzed in the perspective of the bar aspect ratio. That is, as the length of the bar is increased, the mid-sections of the bars experience less tensile stress in the axial direction. The antisymmetric variation of the shearing stress component at section, $y/b = 0.5$, is presented as a function of the bar aspect ratio in Fig.5c. The shearing stresses are also found to be dependent on the aspect ratio of the composite bars, as the maximum and minimum stresses occurred around the stiffened region of the smallest and largest bars, receptively. Both the lateral and axial displacements are also found to decrease when the length of the bar is increased. Therefore, the state of all the displacement and stress components are influenced substantially by the aspect ratio of the stiffened bars, especially, in terms of their magnitude.



(a) Lateral stress component.



(b) Axial stress component.



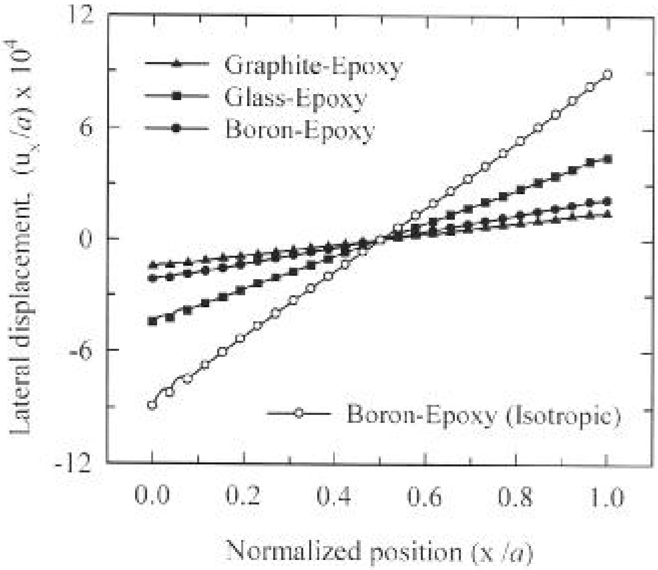
(c) Shear stress component.

Fig.5. Distribution of normalized stress components at section, $y/b = 0.5$, as a function of the aspect ratio of Boron-Epoxy composite bar.

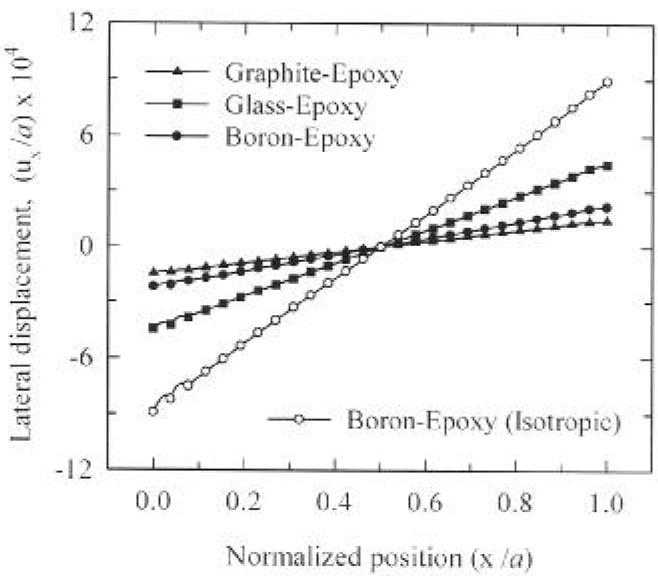
5.2. Effect of material orthotropy on the solution

In an attempt to investigate the effect of material orthotropy, solutions of different displacement and stress components are obtained for different composite materials of interest together with that of the corresponding isotropic material. Following the procedure of Nan *et al.* (1993), the corresponding isotropic material properties for a composite composition of *40%* Boron and *60%* Epoxy resin are obtained as: $\mu = 0.269$ and $E = 2.24 \times 10^6\ psi$. The supporting mathematical treatments required to solve the isotropic stiffened bar problem are given in Appendix-A.

Figure 6 describes the comparison of displacements at the loaded boundary, $y/b = 1$, of the stiffened bars, $b/a = 1$, made of different materials of interest. The general trends of the distributions show that the maximum displacement occurs in Glass-Epoxy and minimum in the Graphite-Epoxy composite bar, when subjected to the same axial tensile loading at the boundary $y/b = 1$. The corresponding displacement level in the Boron-Epoxy bar is found to be inbetween the above two composite materials. Moreover, the displacement level of the isotropic bar is observed to be quite higher than that of the corresponding Boron-Epoxy orthotropic composite bar. Generally, it is expected that the displacement components depend on the stiffness of the composite material in the respective directions. As appears from Tab.1, the elastic modulus is maximum for Graphite-Epoxy and minimum for Glass-Epoxy composites for both the directions of the fiber and matrix. Therefore, the present solution for both the displacement components conforms to the general relation between the strain and stiffness of the composite materials. Further, a quantitative analysis shows that the axial displacement for the isotropic bar increases almost double than that of the corresponding Boron-Epoxy composite bar, while the corresponding increase in lateral displacement is about four times. This may be attributed to the fact that the respective difference in the stiffness along the fiber direction is much higher than that in the transverse direction.



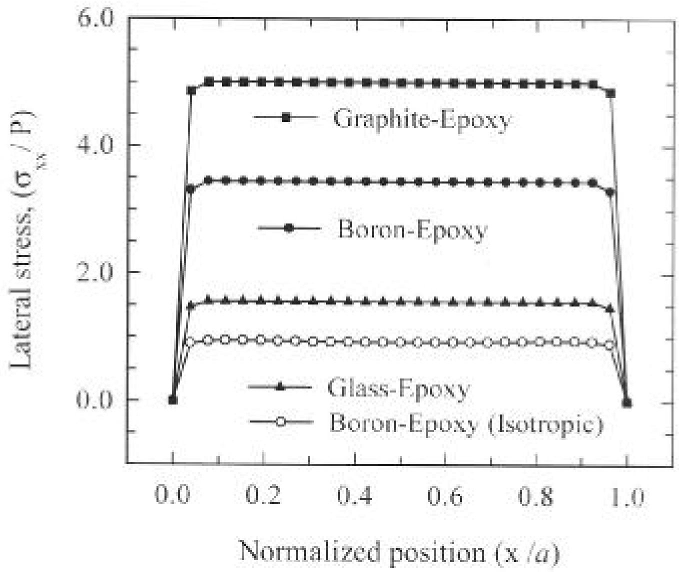
(a) Lateral displacement component.



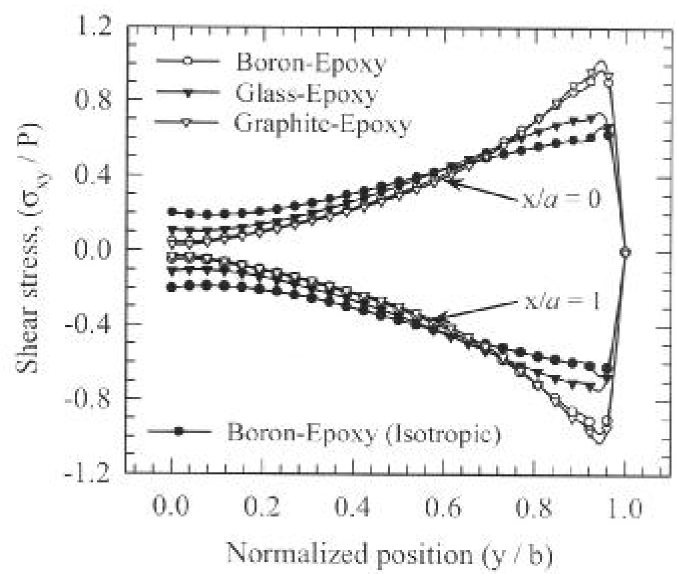
(b) Axial displacement component.

Fig.6. Distribution of normalized displacement components along the right vertical end $(y/b = 1)$ of different composite bars $(b/a = 1)$.

The solutions for the lateral and shear stresses for different composite bars are compared in Fig.7. Figure 7a presents the solution for the stress component, σ_{xx} along the loaded boundary of the stiffened bars $(b/a = 1)$ with different materials of interest. The comparative analysis shows that the maximum stress occurred in the Graphite-Epoxy and minimum stress in the Glass-Epoxy bar. The maximum lateral stress in the Graphite-Epoxy bar is found to be almost five times higher than that of the applied loading. In contrast with that observed with deformation, the lateral stress along the loaded boundary of the Boron-Epoxy composite bar is found to be higher than the corresponding stress in the isotropic bar. Finally, the distribution of the shearing stress along the two stiffened edges of different composite bars $(b/a = 1)$ are presented in Fig.7b. It is interesting to note that, although both the normal stress components vanish along the stiffened boundaries, the shearing stress has got a sharp distribution, which has the maximum value at section very close to the loaded boundary. As appears from the figure, the shearing stress is almost independent of the material used, although the stress level in the Boron-Epoxy composite bar is slightly higher than in the corresponding isotropic bar. It can be noted here that an exact analytical solution of shearing stress along the stiffened boundaries of the composite bar is beyond the capability of the existing mathematical models of elasticity. This limitation has however been successfully removed by the use of the present displacement potential approach.



(a) Lateral stress component.



(b) Shear stress component.

Fig.7. Distribution of normalized stress components along (a) the right vertical end and (b) the stiffened edges of different composite bars $(b/a = 1)$.

6. Conclusions

A new displacement potential approach has been used to analyze the state of stresses and deformations in short flat composite bars with mixed boundary conditions. No appropriate analytical approach was available in the literature which could satisfactorily provide the explicit information about the actual stresses at the critical regions of supports and loadings. Both the qualitative and quantitative results of the present stiffened bar problem of orthotropic composite materials establish the soundness as well as appropriateness of the ψ-formulation. The distinguishing feature of the present single function approach over the existing approaches is that, here, all modes of boundary conditions can be satisfied exactly, whether they are specified in terms of loading or physical restraints or any combination of them; and thus the solutions obtained are promising and satisfactory for the entire regions of interest.

Nomenclature

a, b – dimensions of the bar in *x*- and *y*-directions, respectively
E – elastic modulus of isotropic material
E_f – elastic modulus of fiber material
E_m – elastic modulus of matrix material
E_{11} – elastic modulus of the material in 1-direction
E_{22} – elastic modulus of the material in 2-direction
G – shear modulus of isotropic material
G_{12} – in-plane shear modulus in the 1-2 plane
P – uniformly distributed axial loading on the bar
u_x, u_y – displacement components in the *x*- and *y*-direction
μ – Poisson's ratio of isotropic material
μ_f – Poisson's ratio of fiber material
μ_m – Poisson's ratio of matrix material
μ_{12} – major Poisson's ratio
μ_{21} – minor Poisson's ratio
θ – fiber orientation
σ_{xy} – shearing stress component in the *xy* plane
σ_{xx}, σ_{yy} – normal stress components in the *x*- and *y*-direction
ψ – displacement potential function

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Appendix – A: Solution of the isotropic stiffened bar

The mathematical background for the solution of isotropic stiffened flat bar using the present ψ-formulation is briefly summarized here for ready reference.

Combining Eqs (4.1) and (2.6), the governing ordinary differential equation for the isotropic bar problem is obtained, which is

$$Y_m'''' - 2\alpha^2 Y_m'' + \alpha^4 Y_m = 0 . \tag{A1}$$

The general solution of Eq.(A1) is as follows

$$Y_m = A_m \cosh \alpha y + B_m \alpha y \sinh \alpha y + C_m \sinh \alpha y + D_m \alpha y \cosh \alpha y . \tag{A2}$$

The corresponding expressions for the displacement and stress components in terms of the four arbitrary constants are obtained by combining Eqs (3.3), (4.1), and (A2). Finally, substituting the four boundary conditions, the values of the constants are evaluated following the same procedure used in the solution of the composite bar.

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