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SLIP FLOW IN THE GAS-LUBRICATED RAYLEIGH STEP-SLIDER BEARING

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Singular perturbation methods are applied to analyse the isothermal operation of the Rayleigh step slider bearing of narrow geometry, when the bearing number is moderate and the gas lubricant is rarefied, so that 'slip flow' occurs. Approximations to the pressure field and load-carrying capacity of such a bearing are obtained; and the influence of step geometry and degree of slip on those quantities is discussed.

Key words: Rayleigh step bearing, slip flow, singular perturbations.

1. Introduction

The analysis of the operation of a lubricated slider bearing usually reduces to the problem of determining the pressure distribution in the bearing, its load carrying capacity, and possibly other appropriate design characteristics. In the case of the gas-lubricated bearing operating under isothermal conditions, the governing Reynolds equation is highly nonlinear, and does not permit solution in closed form. In such a case, alternative (approximate) solution techniques must be resorted to.

Frequently, a small parameter may be identified in the problem. Then, perturbation methods based on this parameter may be applied, to construct a closed-form asymptotic expression that approximates the pressure field in some sense, and which may be used to generate similar expressions for other design parameters. Such methods have been applied extensively in the literature.

The Rayleigh step slider bearing is the simplest example of a slider bearing involving a lubricant film profile displaying a discontinuity; i.e., a step. When the lubricant involved is gaseous and rarefied, so-called 'slip flow' occurs, and the Reynolds equation defining the pressure field must be modified somewhat. However, the techniques described above are still applicable.

In the slider bearing, the lubricating pressure is generated by the lateral motion of two surfaces which are not quite parallel. Examples can be seen in flying heads for rotating magnetic memories such as hard disc drives. The steady state pressure distribution and load bearing capabilities of a finite rectangular gaslubricated slider bearing of narrow geometry was investigated by Shepherd and DiPrima (1983). This analysis has been extended to the case where a discontinuity exists in one of the profiles in Penesis et al. (1998). This was also further extended by Penesis (2002). For the cases listed above, non-slip conditions at the bearing boundaries were assumed. However, Schaaf and Sherman (1953) described conditions where slip conditions dominate the boundary layers, and Burgdorfer (1959) used such conditions for the analysis of the Rayleigh step slider in two-dimensional flow.

In the present analysis, the method of matched asymptotic expansions is used, in an approach similar to those noted above, to investigate the Rayleigh step slider bearing in slip flow.

2. Governing equations

Consider a finite rectangular gas step-slider bearing in slip flow of transverse dimension B and longitudinal dimension L, as displayed in Fig.1. Choose coordinates X and Z parallel to L and B, respectively.

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The lower surface (the XZ plane here) moves with constant speed U_0 in the positive X-direction, while the (stationary) upper surface represented by $Y = H(X, Z)$ has a finite transverse jump along the line $X = x_0 L$, for some dimensionless $0 \le x_0 \le 1$. The boundaries are defined by $0 \le X \le L$, $-B/2 \le Z \le B/2$, see Fig.1.

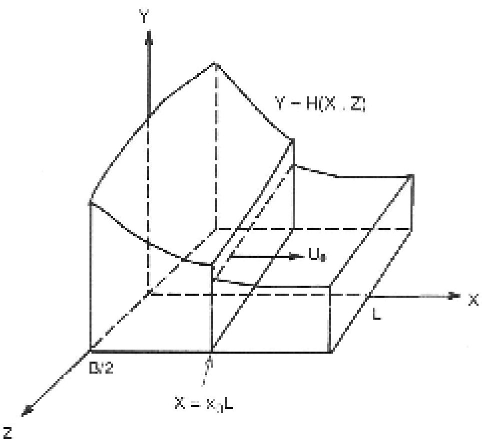

Fig.1. Geometry for the general gas slider bearing with step.

If the lubricating film comprises a gas under isothermal conditions, and slip flow conditions prevail in the bearing gap, the modified Reynolds equation governing the pressure field is given by Burgdorfer (1959)

$$\frac{\partial}{\partial X}\left[PH^3\frac{\partial P}{\partial X}\left(1+\frac{6\lambda_m}{H}\right)\right]+\frac{\partial}{\partial Z}\left[PH^3\frac{\partial P}{\partial Z}\left(1+\frac{6\lambda_m}{H}\right)\right]=6\mu U_0\frac{\partial}{\partial X}(PH) \tag{2.1}$$

where P is the pressure, μ is the gas viscosity and λ_m is the molecular mean free path of the lubricating film. At the bearing boundaries $X = 0$, L, $Z = \pm B/2$, the pressure is assumed to be the ambient, so the boundary conditions there become

$$P(0, Z) = P(L, Z) = P(X, \pm B/2) = P_a \tag{2.2}$$

where P_a is the constant ambient pressure.

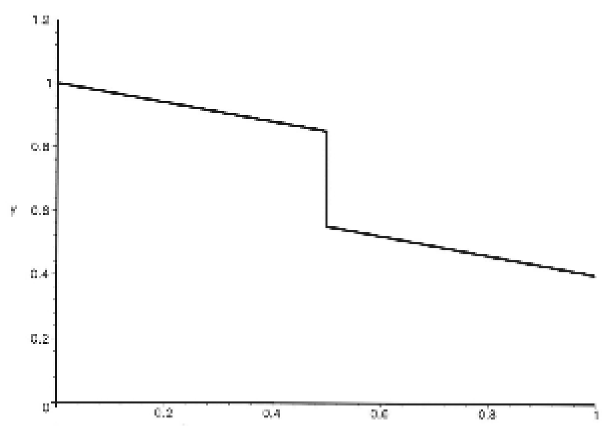

Fig.2. Profile of the wedge step bearing.

3. Dimensionless variables

To non-dimensionalise, define dimensionless variables p, x, z, h by

$$p = \frac{P}{P_a}, \qquad x = \frac{X}{L}, \qquad h = \frac{H}{H_b}, \qquad z = \frac{Z}{B}$$

where P_a is the ambient pressure at the boundaries and H_b is a typical value for the height (e.g., $(H(0, 0))$).

For constant temperature

$$\lambda_m = \frac{\lambda_b}{p}$$

where λ_b is the value of the molecular mean free path at the boundary. Defining the dimensionless parameters ε, Λ, K

$$\frac{B^2}{L^2} = \varepsilon^2, \qquad \Lambda = \frac{6\mu U_0 L}{P_a (H_b)^2}, \qquad K = \frac{6\lambda_b}{H_b}, \tag{3.1}$$

respectively, where ε is the breadth parameter, Λ is the bearing number, and K is the Knudsen number, a measure of the degree of slip in the flow, converts the boundary value problem (2.1), (2.2) to the nonlinear equation

$$\varepsilon^2 \frac{\partial}{\partial x}\left[ph^3 \frac{\partial p}{\partial x}\left(1 + \frac{K}{ph}\right)\right] + \frac{\partial}{\partial z}\left[ph^3 \frac{\partial p}{\partial z}\left(1 + \frac{K}{ph}\right)\right] = \varepsilon^2 \Lambda \frac{\partial}{\partial x}(ph), \tag{3.2}$$

valid on $0 < x < 1$, $-\frac{1}{2} < z < \frac{1}{2}$, together with the boundary conditions

$$p(x, \pm 1/2) = 1, \qquad 0 \le x \le 1, \tag{3.3}$$

$$p(0, z) = p(1, z) = 1, \qquad -1/2 \le z \le 1/2. \tag{3.4}$$

4. Decomposition into two domains

The pressure depends on all of x, z, Λ, K, and ε. For the case considered here, in which the bearing is narrow and Λ and K are moderate, a perturbation approach based on $\varepsilon \to 0$ may be used, so that the Λ-dependence and K-dependence will not be displayed explicitly, and p will simply be written as $p(x, z, \varepsilon)$. The upper bearing surface will be viewed as consisting of the union of two smooth surfaces defined on separate domains

$$0 \le x \le x_0, \qquad -1/2 \le z \le 1/2,$$

and

$$x_0 \le x \le 1, \qquad -1/2 \le z \le 1/2.$$

Clearly, the boundary conditions at the exterior bearing boundaries still hold, while suitable conditions along the common boundary $x = x_0$ are required.

Since the pressure is to be continuous there

$$p(x_0-, z, \varepsilon) = p(x_0+, z, \varepsilon) = \pi(z, \varepsilon) \quad \text{for all} \quad \varepsilon > 0, \quad -1/2 \le z \le 1/2 \tag{4.1}$$

where $\pi(z, \varepsilon)$ is a function to be determined.

The other requirement is that the mass flow across $x = x_0$ be continuous. This is obtained by integrating the partial differential Eq.(3.2) longitudinally across $x = x_0$ and applying Eq.(4.1), to obtain the equation

$$\left[h^3(x, z)\left(1 + \frac{\mathrm{K}}{h(x, z)p(x, z, \varepsilon)}\right)\frac{\partial p(x, z, \varepsilon)}{\partial x} - \Lambda h(x, z)\right]_{x_0-}^{x_0+} = 0\,. \tag{4.2}$$

This condition will be used later to determine a value of the function $\pi(z, \varepsilon)$.

Thus, on the first (leading) domain, it is assumed that the pressure $p(x, z, \varepsilon)$ is the solution of Eq.(3.2) that satisfies the boundary conditions

$$p(x, \pm 1/2, \varepsilon) = 1\,, \qquad 0 \le x \le x_0\,,$$

with

$$\left.\begin{aligned} &p(0, z, \varepsilon) = 1 \\ &p(x_0-, z, \varepsilon) = \pi(z, \varepsilon) \end{aligned}\right\} \quad \text{for} \quad -1/2 \le z \le 1/2\,,$$

while in the second (trailing) domain, it is that solution satisfying

$$p(x, \pm 1/2, \varepsilon) = 1\,, \qquad x_0 < x \le 1\,,$$

with

$$\left.\begin{aligned} &p(x_0+, z, \varepsilon) = \pi(z, \varepsilon) \\ &p(1, z, \varepsilon) = 1 \end{aligned}\right\} \quad \text{for} \quad -1/2 \le z \le 1/2\,.$$

Perturbation methods based on $\varepsilon \to 0$ are now applied to obtain representations for the pressure field in the leading and trailing sections of the bearing. Condition (4.2) is then used to construct the function $\pi(z, \varepsilon)$.

Since $\pi(z, \varepsilon)$ is one of the unknown quantities, it is proposed that

$$\pi(z, \varepsilon) = \pi_0(z) + \varepsilon\pi_1(z) + \varepsilon^2\pi_2(z) + \ldots \tag{4.3}$$

where $\pi_0(z)$, $\pi_1(z)$, ... are to be determined.

5. Perturbation analysis away from the step

For the pressure in the leading bearing section, propose, for $0 < \varepsilon << 1$, the expansion (motivated by only even powers of ε in Eq.(3.2))

$$p(x, z, \varepsilon) = p_0(x, z) + \varepsilon^2 p_2(x, z) + \varepsilon^4 p_4(x, z) + \dots . \tag{5.1}$$

Substituting the expansion (5.1) into Eq.(3.2) and equating like powers of ε yields (to leading order)

$$\frac{\partial}{\partial z}\left[h^3 p_0\left(1 + \frac{\mathrm{K}}{hp_0}\right)\frac{\partial p_0}{\partial z}\right] = 0, \tag{5.2}$$

which must satisfy

$$p_0(x, \pm 1/2) = 1, \qquad\qquad 0 \le x \le x_0, \tag{5.3}$$

and

$$p_0(0, z) = 1, \quad \text{on} \quad -1/2 \le z \le 1/2 . \tag{5.4}$$

Now, since $p_0(x, z)$ must satisfy the second order differential Eq.(5.2) as well as the conditions given at (5.3) and (5.4), which constitute the entire boundary in the leading section of the bearing, it may be argued that for a physically logical expression for the pressure, the leading term of the expansion must be this ambient pressure value. Thus,

$$p_0(x, z) \equiv 1 . \tag{5.5}$$

Note that, unless $\pi_0(z) = 1$, p_0 given by this will not meet the boundary condition at the step. Equating terms of order ε^2 and using (5.5) then gives the boundary-value problem for p_2 as

$$\begin{aligned}
&\frac{\partial}{\partial z}\left[\left(h^3 + \mathrm{K}h^2\right)\frac{\partial p_2}{\partial z}\right] = \Lambda\frac{\partial h}{\partial x},\\
&p_2\left(x, \pm\frac{1}{2}\right) = 0,\\
&p_2(0, z) = p_2(1, z) = 0 .
\end{aligned} \tag{5.6}$$

The differential equation above may be solved subject to the two boundary conditions at $z = \pm 1/2$, to give

$$p_2(x, z) = \Lambda\left[F_1(x, z) - \frac{F_1(x, 1/2)}{F_2(x, 1/2)}F_2(x, z)\right] \tag{5.7}$$

where

$$F_1(x,z)=\int_{-\frac{1}{2}}^{z}\left[h^{-2}(x,s)[h(x,s)+\mathrm{K}]^{-1}\int_0^s\frac{\partial h}{\partial x}(x,t)dt\right]ds\,,$$

and

$$F_2(x,z)=\int_{-\frac{1}{2}}^{z}h^{-2}(x,s)[h(x,s)+\mathrm{K}]^{-1}ds\,.$$

Note that, in general, $p_2(0,z)$ and $p_2(x_0,z)$ do not vanish, so that the expansion $1+\varepsilon^2 p_2(x,z)$ represents the pressure $p(x,z)$ on $0<x<x_0$, $-1/2\le z\le 1/2$, through terms of $O(\varepsilon^2)$, but fails in a neighbourhood of the (local) leading and trailing edges $x=0$ and $x=x_0$ where boundary layer structures are located. These may be analyzed using a local analysis with local (boundary layer) variables. Standard arguments (see Shepherd and DiPrima (1983); Penesis *et al.* (2000)) show that these layers are of $O(\varepsilon)$ as $\varepsilon\to 0$. Thus, near $x=0$, set $x=\varepsilon\xi$, where ξ is $O(1)$. Under this change, Eq.(3.2) transforms to

$$\frac{\partial}{\partial\xi}\left[P_L h^3\left(1+\frac{\mathrm{K}}{P_L h}\right)\frac{\partial P_L}{\partial\xi}\right]+\frac{\partial}{\partial z}\left[P_L h^3\left(1+\frac{\mathrm{K}}{P_L h}\right)\frac{\partial P_L}{\partial z}\right]=\varepsilon\Lambda\frac{\partial}{\partial\xi}(P_L h) \tag{5.8}$$

where $P_L(\xi,z,\varepsilon)\equiv p(\varepsilon\xi,z,\varepsilon)$. Since the bearing is of narrow geometry, it is to be expected that the ambient pressure will be impressed on the layer region as well as that away from layers; at least to leading order. Thus it is proposed that an expansion in the leading layer at $x=0$ is of the form

$$P_L(\xi,z,\varepsilon)=1+\varepsilon P_{L1}(\xi,z)+\varepsilon^2 P_{L2}(\xi,z)+\ldots. \tag{5.9}$$

Substituting (5.9) into (5.8), and equating like powers of ε, gives, to leading order

$$L_0 P_{L1}=\frac{\partial}{\partial\xi}\left[h^3(0,z)+\mathrm{K}h^2(0,z)\right]\frac{\partial P_{L1}}{\partial\xi}+\frac{\partial}{\partial z}\left[h^3(0,z)+\mathrm{K}h^2(0,z)\right]\frac{\partial P_{L1}}{\partial z}=$$
$$\Lambda\frac{\partial}{\partial\xi}(h(0,z))=0. \tag{5.10}$$

The expansion (5.9) must satisfy the boundary conditions at $x=\xi=0$ and at the edges $z=\pm 1/2$. Therefore

$$P_{L1}(0,z)=P_{L2}(0,z)=0\,,\qquad -1/2\le z\le 1/2\,,$$

$$P_{L1}(\xi,\pm 1/2)=P_{L2}(\xi,\pm 1/2)=0\,,\qquad 0\le\xi<\infty\,.$$

The requirement of the matching condition is that the expansion (5.1), as $x\to 0$, must agree with the expansion (5.9) as $\xi\to\infty$. This gives

$$P_{L1}(\xi,z)\to 0\,,\qquad \text{as}\qquad \xi\to\infty\,,$$

$$P_{L2}(\xi, z) \to p_2(0, z), \qquad \text{as} \qquad \xi \to \infty .$$

From this it is deduced that

$$P_{L1}(\xi, z) \equiv 0 ,$$

while the boundary value problem for P_{L2} is found to be

$$L_0 P_{L2} = \Lambda \frac{\partial}{\partial x}(h(0, z)), \tag{5.11}$$

$$P_{L2}(0, z) = 0 , \qquad P_{L2}(\xi, \pm 1/2) = 0 , \tag{5.12}$$

$$P_{L2}(\xi, z) \to p_2(0, z) \quad \text{as} \quad \xi \to \infty . \tag{5.13}$$

Letting

$$P_{L2}(\xi, z) = p_2(0, z) + v_2(\xi, z),$$

and substituting this into Eq.(5.11), gives the boundary value problem for v_2 as

$$L_0 v_2 = 0 , \tag{5.14}$$

$$p_2(0, z), \tag{5.15}$$

$$v_2(\xi, \pm 1/2) = 0 , \tag{5.16}$$

$$v_2(\xi, z) \to 0 \quad \text{as} \quad \xi \to \infty . \tag{5.17}$$

The method of eigenfunction expansions then gives

$$v_2(\xi, z) = \sum_{n=1}^{\infty} A_n e^{-\xi\sqrt{\lambda_n}} \theta_n(z) \tag{5.18}$$

where

$$A_n = -\int_{-\frac{1}{2}}^{\frac{1}{2}} \left[h^3(0, z) + \mathrm{K}h^2(0, z)\right] p_2(0, z) \theta_n(z) dz , \tag{5.19}$$

and λ_n, $\theta_n(z)$ are the eigenvalues and normalised eigenfunctions of the regular Sturm-Liouville system

$$\begin{gathered} \frac{d}{dz}\left[\left(h^3(0, z) + \mathrm{K}h^2(0, z)\right)\frac{d\theta_n}{dz}\right] + \lambda_n \left[h^3(0, z) + \mathrm{K}h^2(0, z)\right]\theta_n(z) = 0 , \\ \theta_n(\pm 1/2) = 0 . \end{gathered} \tag{5.20}$$

A similar analysis may be applied in the trailing section $x_0 < x \le 1$, away from the step $(x = x_0)$. Again, there is a leading order expansion $1+\varepsilon^2 p_2(x, z)$ away from the layer at $x = 1$, with p_2 given by Eq.(5.7), using the local definition of $h(x, z)$. This is corrected in the layer by a boundary layer expansion $1+\varepsilon^2 \tilde{P}_{T2}(\tilde{\xi}, z)$, where $\tilde{\xi} = (1-x)/\varepsilon$ is the layer variable. Here, tildes denote quantities in the trailing section $x_0 < x \le 1$, and $\tilde{P}_{T2} = p_2(1, z) + \tilde{w}_2(\tilde{\xi}, z)$, where $\tilde{w}_2(\tilde{\xi}, z)$ is given by the expression analogous to Eq.(5.18),

$$\tilde{w}_2(\tilde{\xi}, z) = \sum_{n=1}^{\infty} \tilde{B}_n e^{-\tilde{\xi}\sqrt{\tilde{\mu}_n}} \tilde{\psi}_n(z). \tag{5.21}$$

In the above, $\tilde{\mu}_n$, $\tilde{\psi}_n(z)$ are the eigenvalues and eigenfunctions of a system analogous to Eq.(5.20), while $\tilde{B}_n$ are Fourier coefficients analogous to Eq.(5.19), with $h(1, z)$, $p_2(1, z)$ replacing $h(0, z)$, $p_2(0, z)$ respectively.

6. Analysis in the neighbourhood of the step

In the leading bearing section $(0 < x < x_0)$, we introduce the local (layer) variable

$$\zeta = \left(\frac{x_0 - x}{\varepsilon}\right), \qquad 0 \le \zeta \le \infty,$$

with $p(x_0 - \varepsilon\zeta, z, \varepsilon) \equiv P_T(\zeta, z, \varepsilon)$ which converts the original Eq.(3.2) to

$$\frac{\partial}{\partial \zeta}\left[h^3 P_T\left(1+\frac{K}{P_T h}\right)\frac{\partial P_T}{\partial \zeta}\right] + \frac{\partial}{\partial z}\left[h^3 P_T\left(1+\frac{K}{P_T h}\right)\frac{\partial P_T}{\partial z}\right] = -\varepsilon\Lambda \frac{\partial}{\partial \zeta}(P_T h). \tag{6.1}$$

Here, it is proposed that P_T has the expansion

$$P_T(\zeta, z, \varepsilon) = P_{T0}(\zeta, z) + \varepsilon P_{T1}(\zeta, z) + \varepsilon^2 P_{T2}(\zeta, z) + \ldots, \tag{6.2}$$

and substitution and equating like powers of $\in$ gives, to leading order

$$\frac{\partial}{\partial \zeta} = \left[\left(h^3(x_0, z)P_{T0} + Kh^2(x_0, z)\right)\frac{\partial}{\partial \zeta} P_{T0}\right] +$$
$$+ \frac{\partial}{\partial z}\left[\left(h^3(x_0, z)P_0 + Kh^2(x_0, z)\right)\frac{\partial}{\partial z} P_{T0}\right] = 0,$$

with

$$P_{T0}(\zeta, \pm 1/2) = 1, \tag{6.3}$$

$$P_{T0}(0, z) = \pi_0(z), \tag{6.4}$$

$$P_{T0}(\zeta, z) \to 1 \qquad \text{as} \qquad \zeta \to \infty, \tag{6.5}$$

since $\lim_{\zeta \to \infty} P_{T0}(\zeta, z) = \lim_{x \to x_0} p_0(x, z) = 1.$ (6.6)

If $\pi_0(z) \neq 1$, there will arise $O\left(\varepsilon^{-1}\right)$ terms that cannot be matched (from the mass flow condition). From this we can deduce that

$$\pi_0(z) = 1,$$

and consequently

$$P_{T0}(\zeta, z) \equiv 1.$$

Substituting this result into Eq.(6.1) and equating the first power of ε gives

$$\begin{aligned} L_{x_0} P_{T1} = & \frac{\partial}{\partial \zeta}\left[\left(h^3(x_0-, z) + \mathrm{K}h^2(x_0-, z)\right)\frac{\partial}{\partial \zeta} P_{T1}\right] + \\ & + \frac{\partial}{\partial z}\left[\left(h^3(x_0-, z) + \mathrm{K}h^2(x_0-, z)\right)\frac{\partial}{\partial z} P_{T1}\right] = 0, \end{aligned} \tag{6.7}$$

with

$$P_{T1}(\zeta, \pm 1/2) = 0, \tag{6.8}$$

$$P_{T1}(0, z) = \pi_1(z), \tag{6.9}$$

$$P_{T1}(\zeta, z) \to 0 \qquad \text{as} \qquad \zeta \to \infty \tag{6.10}$$

where Eqs (6.8), (6.9) and (6.10) are found by equating coefficients from Eq.(6.2), and matched with the conditions at the step.

Equation (6.7) with associated boundary conditions (6.8), (6.9) and (6.10) can be solved using the method of eigenfunction expansions, and the solution is

$$P_{T1}(\zeta, z) = \sum_{n=1}^{\infty} C_n e^{-\zeta\sqrt{\kappa_n}} \chi_n(z) \tag{6.11}$$

where

$$C_n = \int_{-\frac{1}{2}}^{\frac{1}{2}} \left[h^3(x_0-, z) + \mathrm{K}h^2(x_0-, z)\right]\pi_1(z)\chi_n(z)dz, \qquad n = 1, 2, 3, \ldots$$

are the Fourier coefficients, while κ_n and $\chi_n(z)$ are the eigenvalues and eigenfunctions of the system

$$\frac{d}{dz}\left[\left(h^3(x_0-, z) + \mathrm{K}h^2(x_0-, z)\right)\frac{d\chi_n}{dz}\right] + \kappa_n\left[\left(h^3(x_0-, z)P_0 + \mathrm{K}h^2(x_0-, z)\right)\right]\chi_n(z) = 0,$$

$$\chi_h = (\pm 1/2) = 0\,.$$

In the trailing section, by replacing $P_{T1}(\zeta, z)$, C_n, ζ, κ_n and $\chi_n(z)$ in the above equations with $\tilde{P}_{L1}(\tilde{\zeta}, z)$, D_n, $\tilde{\zeta}$, $\tilde{\nu}_n$ and $\tilde{\phi}_n(z)$ respectively, the counterparts of the leading sections variables, gives the solution for the pressure in the region immediately after the step.

7. The wedge step bearing

The wedge step bearing is a simplification of the upper surface in the geometry. Here, the height does not vary in the transverse direction, but only in the lateral direction, *i.e.*

$$h(x, z) = h(x).$$

In this case, Eq.(5.7) becomes

$$p_2(x, z) = \frac{\frac{1}{2}\Lambda h' x}{h^3 + \mathrm{K}h^2}\left(z^2 - \frac{1}{4}\right). \tag{7.1}$$

This gives the result that

$$v_2(\xi, z) = \sum_{n=1}^{\infty} A_n e^{-n\pi\xi}\theta_n(z) \tag{7.2}$$

where

$$\theta_n(z) = \begin{cases} \left(\dfrac{\sqrt{2}}{h^3(0) + \mathrm{K}h^2(0)}\right)\cos(n\pi z) & n = 1, 3, 5, \ldots.., \\[2ex] \left(\dfrac{\sqrt{2}}{h^3(0) + \mathrm{K}h^2(0)}\right)\sin(n\pi z) & n = 2, 4, 6, \ldots.., \end{cases}$$

and then

$$A_n = -\int_{-\frac{1}{2}}^{\frac{1}{2}} \left[h^3(0) + \mathrm{K}h^2(0)\right] p_2(0, z)\theta_n(z)dz\,, \qquad n = 1, 2, 3, \ldots$$

In the trailing section, $\tilde{\psi}_n(z)$ and $\tilde{B}_n$ are easily found using the expressions for $\theta_n(z)$ and A_n, evaluated at $x = 1$. Similarly, $\chi_n(z)$ and C_n may be found evaluating at $x_0 -$, while $\tilde{\phi}_n(z)$ and $\tilde{D}_n$ are found by using the expressions for $\theta_n(z)$ and A_n and evaluating at $x_0 +$. For the wedge step bearing, $\tilde{\mu}_n = \lambda_n = \tilde{\nu}_n = \kappa_n$.

Now, in the leading section near the step, the solution for P_{T1} is

$$P_{T1}(\zeta, z) = \sum_{n=1}^{\infty} C_n e^{-n\pi\zeta} \chi_n(z) \tag{7.3}$$

where

$$\chi_n(z) = \begin{cases} \left(\dfrac{\sqrt{2}}{h^3(x_{0-}) + \mathrm{K} h^2(x_{0-})} \right) \cos(n\pi z) & n = 1, 3, 5, \ldots, \\ \\ \left(\dfrac{\sqrt{2}}{h^3(x_{0-}) + \mathrm{K} h^2(x_{0-})} \right) \sin(n\pi z) & n = 2, 4, 6, \ldots, \end{cases} \tag{7.4}$$

$$C_n = \int_{-\frac{1}{2}}^{\frac{1}{2}} \left[h^3(x_0 -) + \mathrm{K} h^2(x_0 -) \right] \pi_1(z) \chi_n(z) dz\,, \qquad n = 1, 2, 3, \ldots .$$

8. The Rayleigh step

In this section, the case where $H(x, z)$ is a piecewise constant is considered, see Fig.3. That is, in dimensionless variables

$$h(x, z) = \begin{cases} 1 & \text{for} \quad 0 \le x \le x_0, \quad -1/2 \le z \le 1/2, \\ \\ m & \text{for} \quad x_0 \le x \le 1, \quad -1/2 \le z \le 1/2, \quad \text{where} \quad m = \dfrac{H_1}{H_0} < 1. \end{cases} \tag{8.1}$$

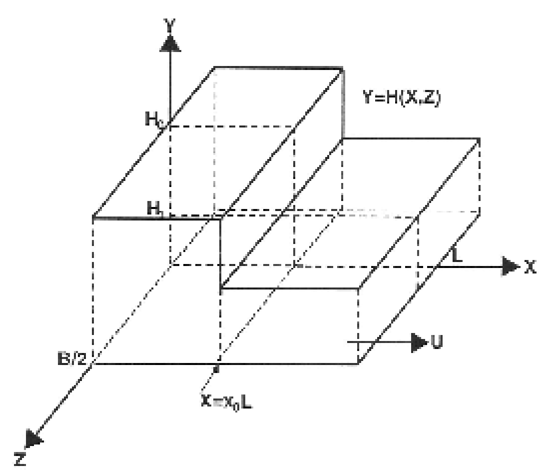

Fig.3. Geometry for the Rayleigh step gas slider bearing.

The expression for $p_2(x, z, \varepsilon)$ is

$$p_2(x, z) = \frac{\frac{1}{2}\Lambda h'(x)}{h^3 + \mathrm{K}h^2}\left(z^2 - \frac{1}{4}\right) = 0,$$

since $h'(x) = 0$ for $0 \le x \le x_0$, $x_0 \le x \le 1$. Since $p_0(x, z, \varepsilon) \equiv 1$, which meets the boundary conditions required at the loading and trailing edges, the pressure at the leading and trailing edges (away from step) is represented, to all orders of ε, by the one term expansion

$$p(x, z, \varepsilon) \equiv 1.$$

The overall (1 term) expansion and the local layer expansion (at the step) can now be combined to give

$$1 + \varepsilon P_{T1}\left(\frac{|x - x_0|}{\varepsilon}, z\right) + O(\varepsilon^2),$$

which is uniformly valid over the whole bearing. Here, P_{T1} is given by Eq.(7.3).

The mass flow condition, given in Eq.(4.2) becomes, in this case

$$\left(1 + \frac{\mathrm{K}}{p}\right)\frac{\partial p}{\partial x}(x_{0-}, z, \varepsilon) - \Lambda = m^3\left(1 + \frac{\mathrm{K}}{mp}\right)\frac{\partial p}{\partial x}(x_{0+}, z, \varepsilon) - \Lambda m.$$

With some simplifications, the mass flow condition becomes

$$\left[\left(1 + \frac{\mathrm{K}}{p(x_0 -)}\right) + m^3\left(1 + \frac{\mathrm{K}}{mp(x_0 +)}\right)\right]\sum_{n=1}^{\infty} n\pi C_n \chi_n(z) = \Lambda(1 - m),$$

or

$$\frac{\Lambda(1 - m)}{\left(1 + \frac{\mathrm{K}}{\pi(z)}\right)(1 + m^3)} = \sum_{n=1}^{\infty} n\pi C_n \chi_n(z) + O(\varepsilon^2).$$

To leading order, $\pi(z) = 1$, and so the mass flow condition is finally

$$\frac{\Lambda(1 - m)}{(1 + \mathrm{K})(1 + m^3)} = \sum_{n=1,3,5}^{\infty} \sqrt{2} n\pi C_n \cos(n\pi z) + \sum_{n=2,4,6}^{\infty} \sqrt{2}\pi\pi, C_n \sin(n\pi z) +$$
$$+ \text{term vanishing} \quad \text{as} \quad \varepsilon \to 0.$$

Notice that the left hand side is an even function, and so it follows that

$$n\pi C_n = \int_{-\frac{1}{2}}^{\frac{1}{2}} \sqrt{2}\cos(n\pi z)dz, \qquad 1 = 1, 3, 5, \ldots.$$

So

$$C_n = \begin{cases} \left(\dfrac{2\sqrt{2}\Lambda(1-m)(-1)^{\frac{n-1}{2}}}{n^2\pi^2(1+\mathrm{K})(1+m^3)} \right), & n = 1, 3, 5, ..., \\ 0, & n = 2, 4, 6, ... \end{cases} \tag{8.2}$$

As stated above, the pressure away from the step is represented by the oneterm expansion. For any improvement in accuracy, higher order terms in the layer must be found.

Equation (8.2) can be used in conjunction with Eq.(7.3) along with the expression for χ_n for a uniform pressure throughout the entire Rayleigh step slider bearing, given by

$$p_u(x, z, \varepsilon) = 1 + \varepsilon \frac{4\Lambda(1-m)}{\pi^2(1+\mathrm{K})(1+m^3)} \sum_{k=0}^{\infty} \frac{(-1)^k}{(2k+1)^2} \cos[(2k+1)\pi z] e^{-(2k+1)\pi|x-x_0|/\varepsilon} + ... \tag{8.3}$$

9. The load

The equation given at Eq.(8.3) can now be used to calculate the load carrying capabilities of the bearing. The dimensionless load, *W*, is given by

$$W = \int_{-\frac{1}{2}}^{\frac{1}{2}} \int_0^1 [p(x,z) - 1] dx\, dz . \tag{9.1}$$

By using the expression for $p_u(x, z, \in)$ found in Eq.(8.3), an approximation to the load is obtained

$$W = \int_{-\frac{1}{2}}^{\frac{1}{2}} \int_0^1 \varepsilon \frac{4\Lambda(1-m)}{\pi^2(1+\mathrm{K})(1+m^3)} \sum_{k=0}^{\infty} \frac{(-1)^k}{(2k+1)^2} \cos[(2k+1)\pi z] e^{-(2k+1)\pi|x-x_0|/\varepsilon} dx\, dz .$$

The order of integration can be rearranged to give

$$W = \varepsilon^2 \frac{16\Lambda(1-m)}{\pi^4(1+\mathrm{K})(1+m^3)} \sum_{k=0}^{\infty} \frac{1}{(2k+1)^4} + ... ,$$

and using the fact that

$$\sum_{k=0}^{\infty} \frac{1}{(2k+1)^4} = \frac{\pi^4}{96} ,$$

this becomes

$$W = \frac{\varepsilon^2 \Lambda(1-m)}{6(1+\mathrm{K})(1+m^3)} + O(\varepsilon^3).$$

10. Discussion

Equation (8.3) makes use of Eqs (8.2), (7.3) and (7.4) to provide a simple procedure for calculating the pressure distribution in the Rayleigh step slider bearing in slip flow. In this case, the calculations are very straightforward, with no boundary layers occurring at the leading and trailing edges. In each case, the approximations are valid for $\Lambda = O(1)$ and $\varepsilon \to 0$.

For the case of the Rayleigh step slider, see Fig.3, the layer at the step is clearly displayed in Fig.4. Here, the excess pressure at the step for varying *z* is shown. The maximum occurs along the midline of the bearing, where $z = 0$, and as the pressure is measured closer to the boundary at the step, the maximum decreases. Figure 5 shows a surface plot of the pressure, which clearly shows the presence of a boundary layer at the step. This agrees with Fig.4, showing that the maximum pressure occurs along the midline of the bearing, where $z = 0$.

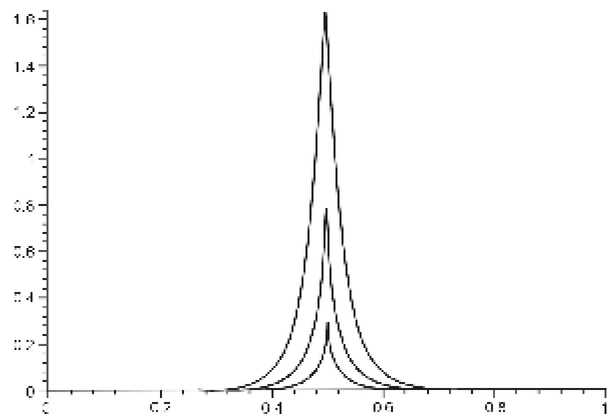

Fig.4. One-dimensional plot of the excess pressure for varying *z*, with $m = \frac{1}{2}$, $x_0 = \frac{1}{2}$, $\varepsilon = 0.1$, $K = 1$ and $\Lambda = 20$.

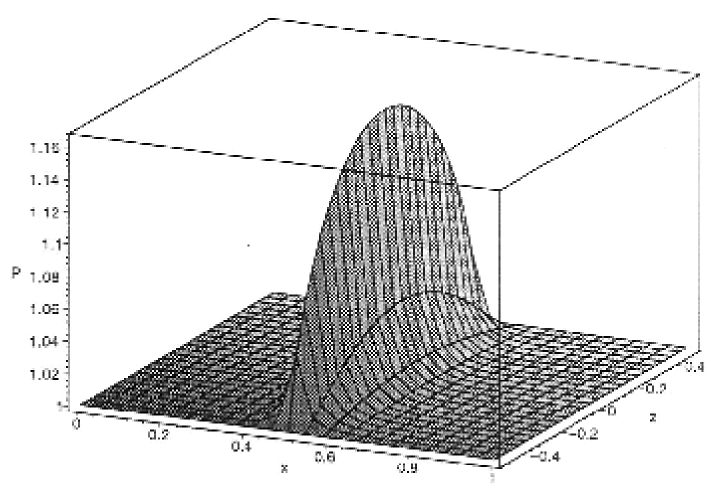

Fig.5. Surface plot of the pressure for the Rayleigh step slider bearing, with $m = \frac{1}{2}$, $x_0 = \frac{1}{2}$, $\varepsilon = 0.1$, $K = 1$ and $\Lambda = 20$.

The graph shown in Fig.6 shows how the load carrying capabilities of the Rayleigh step-slider bearing change with the Knudsen number. As the level of slip in the flow increases, there is a decrease in the maximum load that the bearing is capable of sustaining.

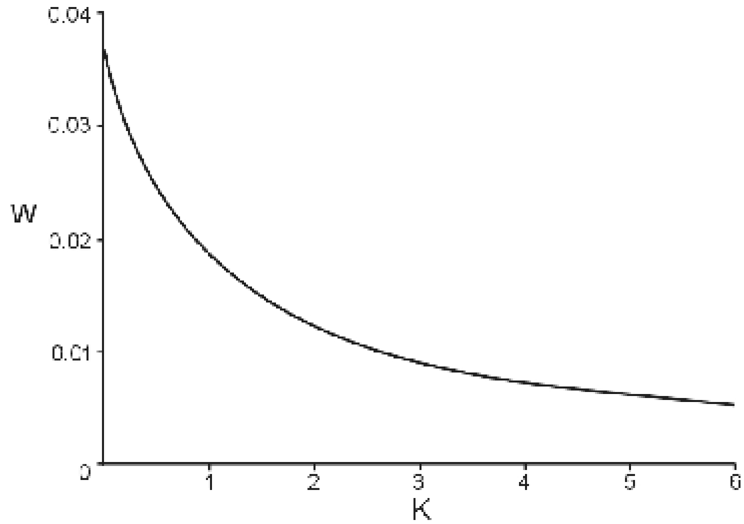

Fig.6. Variation of scaled load *W* with Knudsen number K.

The approximate expressions obtained in previous sections, based on $\varepsilon \to 0$, provide simple methods to estimate the pressure field in a bearing in slip flow, and which may then be used to generate similar expressions for other design parameters.

For a given ε in a Rayleigh step slider, *i.e.*, given breadth to length ratio, and also given operating conditions, the maximum pressure will only depend on the step height variance, *m*. For a larger step, the maximum pressure increases. This is also true for the load of the bearing, which for given ε, K and Λ is proportional to

$$\frac{1-m}{1+m^3}.$$

As the step height increases, the load carrying capabilities of the bearing decrease.

The results of this investigation support those found for the cases where non-slip flows were assumed, in both the presence of boundary layers and in the load capabilities. The validity of the influence that slip flow has on these features has been demonstrated, and must be factored into the solutions.

Nomenclature

$A_n, \tilde{B}_n, C_n$ – Fourier coefficients in leading and trailing bearing sections (Eqs (5.19), (6.11))

H, h – dimensional, dimensionless bearing gap profile function

H_b – typical value of H

K – Knudsen number (Eq.(3.1))

L, B – longitudinal and transverse bearing dimensions

P, p – dimensional, dimensionless bearing gap pressure function

P_a – constant ambient pressure

$P_L, P_T, \tilde{P}_L, \tilde{P}_T$ – pressures in layers in leading and trailing bearing sections

p_u – leading order uniformly valid pressure approximation (Eq.(8.3))

$p_0, p_2, ...$ – pressure components away from layers
U_0 – constant speed of bearing lower surface
W – dimensionless load (Eq.(9.1))
$v_2, \tilde{w}_2$ – layer corrections in leading and trailing bearing sections
x, z – dimensionless X, Z
x_0 – dimensionless location of step
X, Z – longitudinal and transverse bearing space variables
μ – gas lubricant viscosity
λ_m – molecular mean free path
ε – breadth parameter (Eq (3.1))
Λ – bearing number (Eq (3.1))
$\pi, \pi_0, \pi_1, ...$ – pressure and pressure components along step (Eq.(4.3))
$\xi, \tilde{\xi}, \zeta, \zeta$ – local variables in layers in leading and trailing bearing sections
$\theta_n, \tilde{\psi}_n, \chi_n$ – layer eigenfunctions in leading and trailing bearing sections
$\lambda_n, \tilde{\mu}_n, \kappa_n$ – layer eigenvalues in leading and trailing bearing sections

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