SIMILIARITY SOLUTIONS FOR THE UNSTEADY BOUNDARY LAYER FLOW AND HEAT TRANSFER DUE TO A STRETCHING SHEET

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A similarity analysis is presented to investigate the unsteady boundary layers over a stretching sheet for special distributions of the stretching velocity and surface temperature or surface heat flux. The governing unsteady boundary layer equations are reduced to ordinary differential equations with two parameters, the Prandtl number and the unsteadiness parameter. These equations are solved numerically for some values of the governing parameters using the Keller-box method. Some flow and heat transfer characteristics are determined and discussed in detail.

Key words: stretching sheet, unsteady flow and heat transfer**,** boundary layer, numerical results.

1. Introduction

The flow and heat transfer of a viscous and incompressible fluid induced by a continuously moving or stretching surface in a quiescent fluid is relevant to many manufacturing processes. A number of technical processes concerning polymers involves the cooling of continuous strips or filaments by drawing them through a quiescent fluid. Further, glass blowing, continuous casting of metals and spinning of fibres involve the flow due to a stretching surface. In these cases the properties of the final product depend to a great extent on the rate of cooling which is governed by the structure of the boundary layer near the moving strip. Crane (1970) seemed to initiate the study of boundary layer flow due to a stretching surface in an otherwise ambient fluid. He gave a similarity solution in a closed analytical form for the steady boundary layer flow by stretching of a sheet which moves in its own plane with a velocity varying linearly with the distance from a fixed point. Since then several authors have studied various aspects of this problem. Carragher and Crane (1982) investigated the heat transfer in the flow over a stretching surface in the case when the temperature difference between the surface and the ambient fluid is proportional to a power of distance from the fixed point. The temperature field in the flow over a stretching surface subject to a uniform heat flux was studied by Dutta *et al*. (1985), and Grubka and Bobba (1985), while Elbashbeshy (1998) considered the case of a stretching surface with a variable surface heat flux. Lin and Chen (1998) presented an exact solution of heat transfer from a stretching surface with a variable heat flux. Gupta and Gupta (1977) examined the heat and mass transfer for the boundary layer flow over a stretching sheet subject to suction and blowing. The effects

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of power law surface temperature and power law surface heat flux on the heat transfer characteristics of a continuous stretching surface with suction and blowing were investigated by Chen and Char (1988). Magyari and Keller (1999, 2000) obtained analytical solutions for the case when the sheet is permeable and also for the case when the velocity and temperature of the sheet varies exponentially with the distance along the sheet. Liao and Pop (2004) have recently studied the problem of a steady boundary layer flow due to a stretching sheet using the Homotopy Analysis Method (HAM) proposed by Liao (2003) and obtained analytic solutions of the flow characteristics.

The above studies deal with a steady flow only. However, in some cases the flow field and heat transfer can be unsteady due to a sudden stretching of the flat sheet or by a step change of the temperature or heat flux of the sheet. When the surface is impulsively stretched with certain velocity, the inviscid flow is developed instantaneously. However, the flow in the viscous layer near the sheet is developed slowly, and it becomes a fully developed steady flow after a certain instant of time. The flow problem caused by the impulsive stretching of a sheet has been investigated by a number of authors (Na and Pop, 1996; Wang ***et al.***, 1997 and Nazar ***et al.***, 2004). Recently, Elbashbeshy and Bazid (2004) have presented similarity solutions of the boundary layer equations, which describe the unsteady flow and heat transfer over a stretching sheet. The governing unsteady boundary layer equations are transformed to ordinary differential equations by considering the velocity and the temperature of the stretching sheet of a particular form. The governing similarity equations contain only the Prandtl number and the unsteadiness parameter. Although a similarity solution is accomplished by these authors, some physically unrealistic phenomena are encountered for specific values of the unsteadiness parameter.

The present analysis aims to study the unsteady flow and heat transfer over a stretching sheet in a viscous and incompressible fluid which is at rest under the similarity conditions considered by Elbashbeshy and Bazid (2004). In addition, both the variable wall temperature (VWT) and variable heat flux (VHF) conditions have been considered. The governing equations are solved numerically using a very efficient finite-difference method known as Keller-box method.

2. Basic equations

Consider the unsteady flow and heat transfer of a viscous and incompressible fluid past a semi-infinite stretching sheet in the region $y > 0$, as shown in Fig.1.

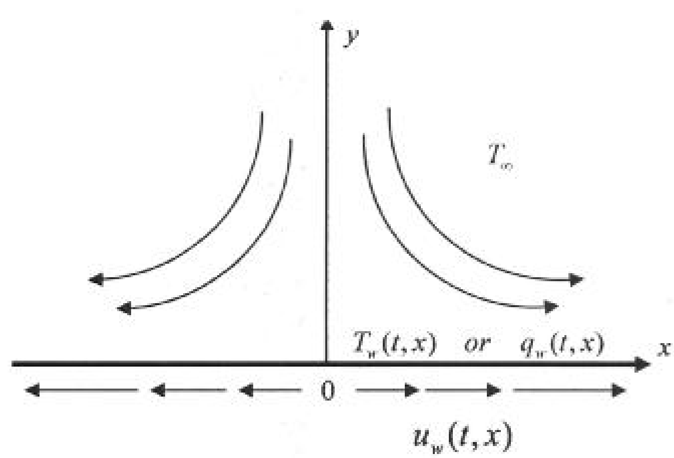

Fig.1. Physical model and coordinate system.

Keeping the origin fixed, two equal and opposite forces are suddenly applied along the $x-$axis, which results in stretching of the sheet and hence, flow is generated. At the same time, the wall temperature $T_w(t,x)$

of the sheet is suddenly raised from T_∞ to $T_w(t,x)(>T_\infty)$ or there is suddenly imposed a heat flux $q_w(t,x)$ at the wall. Under these assumptions, the basic unsteady boundary layer equations governing the flow and heat transfer due to the stretching sheet are given by

$$\frac{\partial u}{\partial x}+\frac{\partial v}{\partial y}=0, \tag{2.1}$$

$$\frac{\partial u}{\partial t}+u\frac{\partial u}{\partial x}+v\frac{\partial u}{\partial y}=\upsilon\frac{\partial^2 u}{\partial y^2}, \tag{2.2}$$

$$\frac{\partial T}{\partial t}+u\frac{\partial T}{\partial x}+v\frac{\partial T}{\partial y}=\alpha\frac{\partial^2 T}{\partial y^2}, \tag{2.3}$$

subject to the initial and boundary conditions

$$\begin{aligned}
&t<0: \quad u=v=0, \quad T=T_\infty \quad \text{for any} \quad x, y, \\
&t\geq 0: \quad u=u_w(t,x), \quad v=0, \\
&\qquad T=T_w(t,x) \quad \text{(VWT)} \quad \text{or} \quad \frac{\partial T}{\partial y}=-\frac{q_w(t,x)}{k} \quad \text{(VHF)}, \\
&\qquad u\to 0, \quad T\to T_\infty \quad \text{as} \quad y\to\infty
\end{aligned} \tag{2.4}$$

where t is the time, u and v are the velocity components along the x- and y- axes respectively, T is the temperature, α is the thermal diffusivity, υ is the kinematic viscosity and k is the thermal conductivity. We assume now that the velocity of the sheet $u_w(t,x)$, the sheet temperature $T_w(t,x)$ and the heat flux $q_w(t,x)$ have the following form

$$\begin{aligned}
&u_w(t,x)=cx(1-\gamma t)^{-1}, \qquad T_w(t,x)=T_\infty+\frac{c}{2\upsilon x^2}(1-\gamma t)^{-3/2}, \\
&q_w(t,x)=\frac{q_{w0}}{2x^2}(c/\upsilon)^{3/2}(1-\gamma t)^{-2}
\end{aligned} \tag{2.5}$$

where c is the stretching rate being a positive constant, γ is a positive constant, which measures the unsteadiness and q_{w0} is a characteristic heat transfer quantity. We introduce now the following new variables

$$\begin{aligned}
&\eta=\sqrt{\frac{c}{\upsilon(1-\gamma t)}}\,y, \qquad \psi=\sqrt{\frac{c\upsilon}{(1-\gamma t)}}\,xf(\eta), \\
&T=T_\infty+\frac{c}{2\upsilon x^2}(1-\gamma t)^{-3/2}\theta(\eta) \quad \text{(VWT)}, \\
&T=T_\infty+(q_{w0}/k)\frac{c}{2\upsilon x^2}(1-\gamma t)^{-3/2}\theta(\eta) \quad \text{(VHF)}
\end{aligned} \tag{2.6}$$

where ψ is the stream function which is defined in the usual way as $u=\partial\psi/\partial y$ and $v=-\partial\psi/\partial x$. Substituting variables Eqs (2.6) into Eqs (2.2) and (2.3), they reduce to the following ordinary differential equations

$$f''' + ff'' - f'^2 - A\left(f' + \frac{1}{2}\eta f''\right) = 0, \tag{2.7}$$

$$\frac{1}{\Pr}\theta'' + f\theta' + 2f'\theta + -A(3\theta + \eta\theta') = 0. \tag{2.8}$$

subject to the boundary conditions (2.4), which become

$$\begin{aligned} &f(0)=0, \qquad f'(0)=1, \qquad \theta(0)=1 \quad \text{(VWT)} \qquad \text{or} \qquad \theta'(0)=-1 \quad \text{(VHF)}, \\ &f'(\infty)=0, \qquad \theta(\infty)=0 \end{aligned} \tag{2.9}$$

where $\Pr$ is the Prandtl number, $A=\gamma/c$ is a non-dimensional constant which measures the flow and heat transfer unsteadiness and primes denote the differentiation with respect to the similarity variable η.

The physical quantities of interest in this problem are the skin friction coefficient C_f and the local Nusselt number Nu_x which are defined as

$$C_f = \frac{\tau_w}{\rho u_w^2}, \qquad \mathrm{Nu}_x = \frac{x q_w}{k(T_w - T_w)} \tag{2.10}$$

where the skin friction τ_w and the heat transfer from the sheet q_w are given by

$$\tau_w = \mu\left(\frac{\partial u}{\partial y}\right)_{y=0}, \qquad q_w = -k\left(\frac{\partial T}{\partial y}\right)_{y=0}, \tag{2.11}$$

with μ being the dynamic viscosity. Using variables Eqs (2.5) and (2.6), we get

$$\begin{aligned} &C_f \mathrm{Re}_x^{1/2} = f''(0), \qquad \mathrm{Nu}_x/\mathrm{Re}_x^{1/2} = -\theta'(0) \quad \text{(VWT)}, \\ &\mathrm{Nu}_x/\mathrm{Re}_x^{1/2} = \frac{1}{\theta(0)} \quad \text{(VHF)} \end{aligned} \tag{2.12}$$

where $\mathrm{Re}_x = u_w x/\upsilon$ is the local Reynolds number.

3. Results and discussion

Equations (2.7) and (2.8) together with the boundary conditions form a nonlinear two-point boundary value problem, which has been solved numerically using a very efficient implicit finite-difference method known as Keller-box method, which is discussed by Cebeci and Bradshaw (1984). Results are given for some values of the unsteady parameter A and the Prandtl number $\Pr$. The accuracy of this numerical method was validated for the case of (VWT) by a direct comparison with the numerical results reported by Grubka and Bobba (1985) and Elbashbeshy and Bazid (2004) for the steady-state flow case $(A=0)$ and $\Pr = 1.0$. Table 1 presents results of this comparison for the heat transfer from the sheet, $-\theta'(0)$. The value of the surface temperature, $\theta(0)$, for the case of (VHF) is also included in this table. It can be seen from this table that a very good agreement between the results exists.

Table 1. Values of the heat transfer $-\theta'(0)$ for $A = 0.0$ (steady-state flow) and $\Pr = 1.0$.

Present	Grubka and Bobba (1985)	Elbashbeshy and Bazid (2004)
0.99999	*1.00000*	*0.99999*

Figures 2 to 4 show the variation of the velocity and temperature profiles $f'(\eta)$ and $\theta(\eta)$ with η for several values of the parameter A and the Prandt number Pr. It can be seen from Fig.2 that the velocity profiles decrease continuously to zero with the increase of the parameter A without any flow reversal. This is contrary to the results of Fig.1 from the paper by Elbashbeshy and Bazid (2004) where there is a flow reversal for all values of $A \neq 0$ considered. Further, Fig.3 show that the temperature profiles also decrease monotonously with the increase of A, except for $A = 0$. In this case $(A = 0)$ the temperature profiles overshoot its value at the surface of the sheet, as can be seen from Fig.3a. This behaviour is in agreement with the results of Grubka and Bobba (1985) but not with that of Elbashbeshy and Bazid (2004). However, we notice from Fig.3b, which, correspond to the case of (VHF) that for $A = 0$ (steady-flow case) the non-dimensional temperature profile is completely negative within the thermal boundary layer. This negative profile is a physically unrealistic case. It implies the violation of the second law of thermodynamics where heat transfer cannot proceed from a lower temperature to a higher temperature. Finally, the effect of Pr on the non-dimensional temperature profile is illustrated in Fig.4 for some values of Pr and a fix value of $A = 0.8$. These profiles decrease, while the surface heat transfer increases with Pr, as can be seen from Tab.2. The physical reason for this trend is that a higher Prandtl number fluid has a thinner thermal boundary layer which increases the gradient of the temperature. Consequently, the surface heat transfer increases as Pr increases. It is also noticed from Tab.2 that for the (VHF) case the wall temperature decreases with the increase of Pr. However, it can be observed from Tab.2 that there are some discrepancies between the present results and those of Elbashbeshy and Bazid (2004), which might lead to the above mentioned reversed flow.

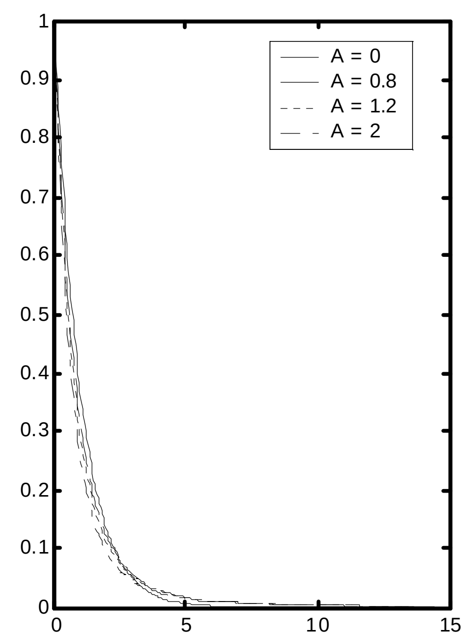

Fig.2. Velocity profiles for Pr=*1* and several values of *A*.

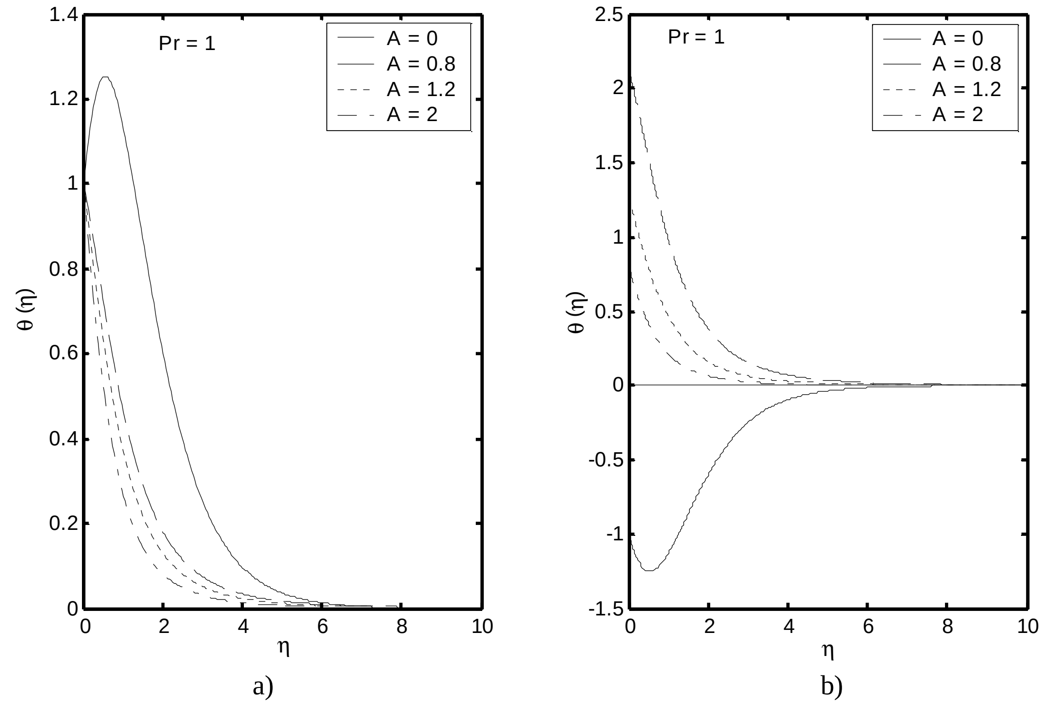

Fig.3. Temperature profiles for Pr=*1* and several values of *A*. a) Variable wall temperature (VWT); b) Variable heat flux (VHF).

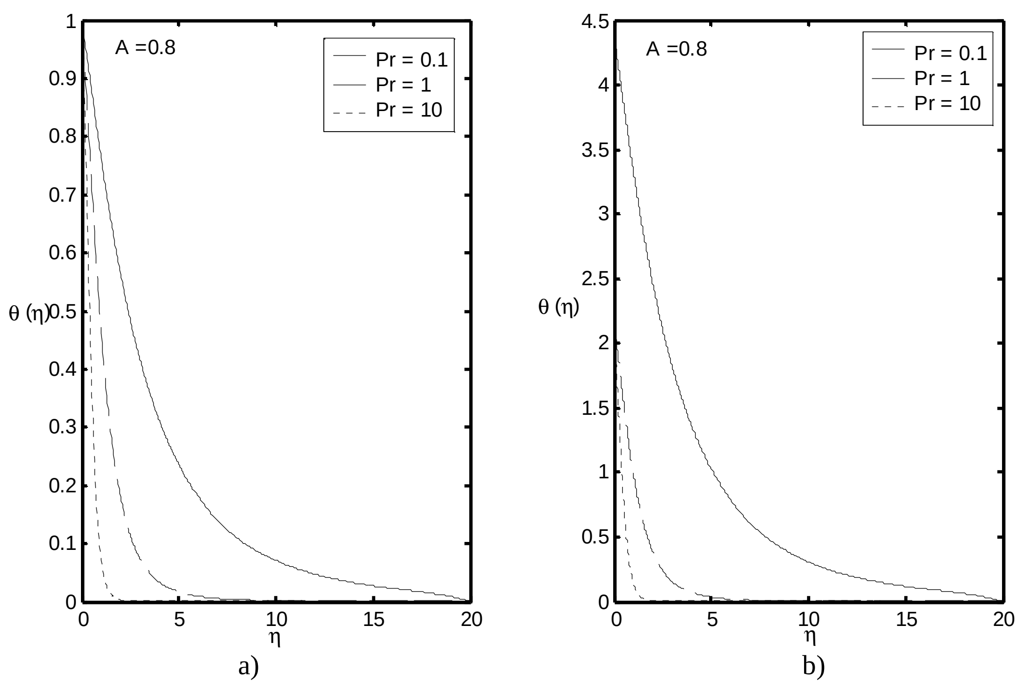

Fig.4. Temperature profiles for *A=0.8* and several values of Pr. a) Variable wall temperature (VWT); b) Variable heat flux (VHF).

Table 2. Values of the heat transfer $-\theta'(0)$, temperature on the wall $\theta(0)$ and skin friction coefficient $-f''(0)$, for various values of A and Pr.

A	0.8			1.2			10		
Pr	$-\theta'(0)$	$\theta(0)$	$-f''(0)$	$-\theta'(0)$	$\theta(0)$	$-f''(0)$	$-\theta'(0)$	$\theta(0)$	$-f''(0)$
0.01	*0.092274 (0.1016)*	*10.837316*	*1.261042 (1.3321)*	*0.114053 (0.1319)*	*8.767842*	*1.377722 (1.4691)*	*0.150317 (0.1723)*	*6.652599*	*1.587362 (1.7087)*
0.1	*0.229433 (0.2707)*	*4.358565*	*1.261042 (1.3321)*	*0.311720 (0.3576)*	*3.208012*	*1.377722 (1.4691)*	*0.438750 (0.4916)*	*2.79204*	*1.587362 (1.7087)*
1.0	*0.471190 (0.6348)*	*2.12287*	*1.261042 (1.3321)*	*0.788173 (0.9491)*	*1.268756*	*1.377722 (1.4691)*	*1.243741 (1.4086)*	*0.804026*	*1.587362 (1.7087)*
10.0	*0.510385 (1.2552)*	*1.959306*	*1.261042 (1.3321)*	*1.859897 (2.4177)*	*0.537664*	*1.377722 (1.4691)*	*3.547142 (3.9814)*	*0.281917*	*1.587362 (1.7087)*

( ) Elbashbeshy and Bazid (2004)

4. Conclusions

The present study provides similarity solutions for the unsteady laminar boundary layer flow and heat transfer over a stretching sheet. The results show that the non-dimensional velocity profiles are compressed and suppressed toward the sheet with increasing values of the unsteady parameter A. Temperature profiles, on the other hand, become fuller and the surface heat flux increases and the wall temperature considerably decreases with the increase of A. Also, the surface heat transfer increases with increasing Pr causing a decrease in the thermal boundary layer thickness.

Nomenclature

- A – dimensionless measure of the unsteadiness
- c – stretching rate
- C_f – skin friction coefficient
- k – thermal conductivity
- Nu_x – local Nusselt number
- Pr – Prandtl number
- Re_x – local Reynolds number
- q_w – heat flux at the surface of the sheet
- q_{w0} – characteristic wall heat flux
- t – time
- T – fluid temperature
- T_w – surface temperature
- T_∞ – ambient temperature
- u, v – velocity components along $x-$ and $y-$ axes
- u_w – velocity of the moving sheet
- x, y – Cartesian coordinates along the sheet and normal to it respectively
- α – thermal diffusivity of the fluid
- γ – positive constant
- η – similarity variable
- θ – non-dimensional temperature

υ – kinematic viscosity
ρ – density
τ_w – skin friction
ψ – stream function

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