ON THE MATRIX TECHNIQUES FOR MODELING OF THERMOELASTIC WAVES IN LAYERED STRUCTURES

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The development of a general purpose model for thermoelastic wave propagation in a Cartesian system for heat conducting isotropic layered plates is illustrated. The model can account for elastic and visco-elastic isotropic materials, single or multi-layered structures, and free or leaky systems. A theoretical treatment is presented for calculating the displacements, temperature, thermal stresses and temperature gradient within a multilayered plate in generalized theories of thermoelasticity, using the matrix transfer technique. A rigidly bonded and smooth interface condition is also considered as a special case to stimulate de-bonding of two materials layers. The model developed will be of value in material characterization and as a source of other quantitative information on thermo-mechanical, strength related properties of advanced materials. Relevant results of previous investigations and coupled thermoelasticity are deduced as special cases.

Key words: generalized thermoelasticity, layered structures, transfer matrix, dispersion, rigid and smooth interface.

1. Introduction

A layered medium consists of two or more material components connected at their interface in some manner. Use of ultrasonic for nondestructive inspection frequently involves the study of the interaction of sound with multilayered plate structures. The development of inspection techniques for such tasks requires the study of complicated wave mechanics and relies strongly on the use of predictive modeling tools to enable the best inspection strategies to be identified and their sensitivities to be evaluated.

Propagation of elastic waves in layered media (Ewing ***et al***., 1957; Brekhovskikh, 1960; Achenbach, 1973) have long been of interest of researchers in the fields of geophysics, acoustics, and electromagnetic. In thermoelasticity, dynamic problems in layered media turn out to be even more difficult than its counterpart in elasticity, as in thermoelasticity solutions to both the heat conduction and thermoelasticity problems for all the layers are required. These solutions are also to satisfy the thermal and mechanical boundary and interface conditions. As a result, a conventional procedure for thermal stress analysis of a multilayered medium results in having to solve a system of two simultaneous equations for a large number of unknown constants. Thus the study of thermo-mechanical interactions and wave phenomenon in composites materials is justified and is of great importance and practical use in engineering applications, especially in thermoelasticity. Because of dissimilar mechanical and thermal properties between constituents, thermal effects are largely the origin of failure for a wide class of multilayered devices as temperature changes can introduce residual stresses, which may lead to interface de-bonding in ductile materials.

Although Fourier's law of heat conduction is suitable in describing common engineering situations, however, it breaks down in situations involving very short times, high heat fluxes, and at very low temperatures. The anomaly of this classical theory of heat conduction is from the assumption that the heat flux vector and the temperature gradient across a material volume occur at the same instant of time. As a consequence of this assumption, the equation is governed by a parabolic partial differential equation, which predicts that the thermal disturbance in a body instantaneously affects all points of the body. To remove this paradox inherent in the classical theory, a theory of generalized thermoelasticity was developed.

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Lord and Schulman (LS) (1967), and Green and Lindsay (GL) (1972) extended the coupled theory of thermoelasticity by introducing the thermal relaxation time in the constitutive equations. These new theories eliminating the paradox of infinite velocity of heat propagation are called the generalized theories of thermoelasticity. They have received much attention in recent years. The literature dedicated to such theories is quite large and its detailed review can be found in Nowacki (1962; 1975), Chadwick (1960) and Chandrasekharaiah (1986; 1998). The LS model introduces a single time constant to dictate the relaxation of thermal propagation, as well as the rate of change of strain rate and the rate of change of heat generation. In the GL model, on the other hand, the thermal and thermo-mechanical relaxations times are governed by two different time constants.

The works of Ignaczak (1989), Hawwa and Nayfeh (1996), Daimaruya and Naitoh (1987) Green and Naghdi (1991) contain more detailed discussions on this phenomenon. Several authors (Nayfeh and Nasser, 1971; Tao and Prevost, 1984; Massalas and Kalpakidis, 1987a; b) have considered the propagation of generalized thermoelastic waves in plates of isotropic media. Padovan (1974; 1975), Tauchert (1975; 1980), Tanigawa *et al*. (1989), Bufler (1971), Bahar (1972) proposed the transfer matrix method to study the isothermal elasticity problems in a multilayered medium, and later extended it to thermoelasticity by Bahar and Hetnarski (1980). Thangjitham and Choi (1991) extended the flexibility /stiffness matrix method to the thermoelastic problem of a multilayered anisotropic medium under the state of generalized plane deformation.

Nayfeh and Taylor (1990) studied the problem of dynamic distribution of displacement and stress considerations in the ultrasonic immersion nondestructive evaluation of multilayered plates for a plate composed of an arbitrary number of isotropic layers. Verma *et al.* (1999a) studied the dynamic distribution of displacements and thermal stresses in multilayered media in generalized thermoelasticity. Verma and Hasebe (1999b; 2001; 2002) studied the thermoelastic problem in generalized theories of thermoelasticity.

This paper illustrates the development of a general-purpose model from matrix formulations, which describe thermoelastic waves in heat conducting isotropic-layered materials, with an arbitrary number of layers in a Cartesian system. The models developed can account for elastic and visco-elastic isotropic materials, single or multi-layered structures, and free or leaky systems. Although the Global Matrix method is different from the Transfer Matrix Method, which is apparently based on the transfer matrix, both of them will yield the same dispersion results since they are deduced from the same traction-free surfaces and interface continuity conditions. Theoretical calculations of the displacements, temperature, thermal stresses and temperature gradient within a multilayered plate in generalized theories of thermoelasticity, using the matrix transfer technique are also discussed as special cases. A rigidly bonded and smooth interface condition is also considered as a special case to stimulate de-bonding of two material layers. The model developed will be of value in material characterization and providing other quantitative information on thermo-mechanical, strength related properties of advanced materials. Relevant results of previous investigations and coupled thermoelasticity are deduced as special cases.

2. Field equations of theories of generalized thermoelasticity

The generalized coupled field equations governing dynamic thermoelastic processes for homogeneous heat conducting isotropic materials and in the absence of body forces and heat source can be written as

$$\mu\nabla \boldsymbol{u} + (\lambda+\mu)\nabla\nabla \boldsymbol{u} - \gamma\left(1+\tau_1\frac{\partial}{\partial t}\right)\nabla\theta = \rho\ddot{\boldsymbol{u}}, \tag{2.1}$$

$$K\nabla^2\theta + \rho C_e\left[\dot{\theta} + (\tau_2+\tau_0)\ddot{\theta}\right] = \gamma\theta_0\left(\nabla\dot{\boldsymbol{u}} + \tau_0\nabla\ddot{\boldsymbol{u}}\right) \tag{2.2}$$

where $\gamma = (3\lambda + 2\mu)\alpha_t$; λ and μ are the Lame' parameters and α_t is the coefficient of thermal expansion; $\boldsymbol{u}$ is the displacement vector; θ is the temperature change above the uniform reference temperature θ_0; ρ is the mass density; C_e is the specific heat at constant deformation; τ_1, τ_0 and τ_2 are the thermal relaxation times and thermal-mechanical relaxation, respectively. In Lord and Shulman (LS) theory, $\tau_1 = \tau_2 = 0$, with

τ_0 is the thermal relaxation time of heat flux and in Green and Lindsay (GL) theory, $\tau_0 = 0$, and τ_1 and $\tau_2 (= \tau_0)$ are the thermal-mechanical relaxation and thermal relaxation times, respectively.

Consider a plate consisting of an arbitrary number, N, of homogeneous thermoelastic isotropic layers rigidly bonded at their interfaces. The problem is to study the displacements, temperature, thermal stresses and temperature gradient induced within the plate by an incident wave at an arbitrary angle from the normal to the interface. We shall use two sets of two-dimensional coordinate systems (x, z). One system is the global coordinate system, which has its origin at the bottom layer of the plate such that x denotes the propagation direction and z is the normal to the interfaces. Hence a layered plane will then occupy the space $0 \le z \le d$ where d denotes the total thickness of the plate. The second system is local for each sublayer of the plate. Since the plate is made of N layers, the kth layer will then have its local coordinates $x^{(m)}$ and $z^{(m)}$ with a local origin at the bottom surface. Hence each layer occupies the space $0 \le z^{(m)} \le d^{(m)}$ where $d^{(m)}$ is its thickness. With this choice of the coordinate system the equations of motion and heat conduction for each layer are given by Lord and Schulman (1967), and Green and Lindsay (1972)

$$\mu\left(u_{,xx} + u_{,zz}\right) + (\lambda + \mu)\left(u_{,xx} + w_{,xz}\right) = \rho\ddot{u} + \gamma\left(T_{,x} + \tau_1 \dot{T}_{,x}\right), \tag{2.3}$$

$$\mu\left(w_{,xx} + w_{,zz}\right) + (\lambda + \mu)\left(u_{,xx} + w_{,zz}\right) = \rho\ddot{w} + \gamma\left(T_{,z} + \tau_1 \dot{T}_{,z}\right), \tag{2.4}$$

$$K\left[T_{,xx} + T_{,zz}\right] + \rho C_e\left(\dot{T} + \tau_0 \ddot{T}\right) = \gamma T_0\left[\left(\dot{u}_{,x} + \dot{w}_{,z}\right) + \tau_0 \delta_{1k}\left(\ddot{u}_{,x} + \ddot{w}_{,z}\right)\right] \tag{2.5}$$

where the thermal relaxation times satisfy the inequalities $\tau_1 \ge \tau_0 \ge 0$ (for Green-Lindsay theory only), λ and μ are Lame's constants, $\gamma = (3\lambda + 2\mu)\alpha_t$ is the thermoelastic coupling constant and α_t the coefficient of thermal expansion and all other symbols have their usual meanings as in Lord and Schulman (1967), and Green and Lindsay (1972). The comma notation is used for spatial derivatives and the superposed dot denotes time differentiation.

3. Waves in an isotropic thermoelastic flat-layered plate

For waves whose projected wave vector is along the x-axis, Eqs (2.3)-(2.5) admit the formal solutions

$$(u, w, T) = (U_1, U_2, U_3)\exp\left[i\xi(x + \alpha z - ct)\right] \tag{3.1}$$

where ξ is the wave number, U_1, U_2 and U_3 are the constant amplitudes related to displacements and temperature, c is the phase velocity $(= \omega/\xi)$, ω is the circular frequency, α is the ratio of the z and x-directions wave numbers. This choice of solutions leads to the coupled equations

$$M_{mn}(\alpha)U_n = 0, \qquad m,\ n = 1,2,3 \tag{3.2}$$

where the summation convention is implied, and

$$\begin{aligned}
&M_{11} = c_2\alpha^2 + 1 - \zeta^2, \qquad M_{13} = c_3\alpha, \qquad M_{14} = 1, \\
&M_{13} = M_{31}, \qquad M_{33} = c_2 + \alpha^2 - \zeta^2, \qquad M_{34} = \alpha, \\
&M_{41} = i\xi c\omega^* \zeta^2 \tau^\bullet \varepsilon_1, \qquad M_{43} = i\xi c\omega^* \zeta^2 \alpha \tau^\bullet \varepsilon_1, \\
&M_{44} = \omega^* \zeta^2 \tau - \left(1 + \alpha^2\right), \qquad c_3 = 1 - c_2,
\end{aligned} \tag{3.3}$$

$$\varepsilon_1=\frac{T_0\gamma^2}{\rho C_e(\lambda+2\mu)},\qquad \omega^*=C_e(\lambda+2\mu)/K,$$

$$\tau^\bullet=(\tau_0\delta_{1k}+i/\xi c)(\tau_1+i/\xi c),\qquad \tau=(\tau_0+i/\xi c),\tag{3.4}$$

$$c_2=\mu/(\lambda+2\mu),\qquad \zeta^2=\rho c^2/(\lambda+2\mu).$$

The existence of a nontrivial solution for U_1, U_2 and U_3 demands the vanishing of the determinant in Eq.(3.2), and yields the polynomial equation

$$\left(\zeta^2-c_2\alpha^2-c_2\right)\left(\alpha^4+A\alpha^2+B\right)=0\tag{3.5}$$

where

$$A=2-\left[\left(\omega^*\tau+1\right)-i\xi c\tau^\bullet\varepsilon_1\right]\zeta^2,$$

$$B=\left[\left(\zeta^2-1\right)\left(\zeta^2\omega^*\tau-1\right)+i\xi c\zeta^2\tau^\bullet\varepsilon_1\right].$$

Solving (3.5) for the six roots of α and using superposition results in the following formal solution relating the displacements, temperature, thermal stresses and temperature gradient within a layer to its wave amplitudes.

$$\begin{bmatrix}u\\ w\\ T\\ \breve{\sigma}_{zz}\\ \breve{\sigma}_{xz}\\ \breve{T}'\end{bmatrix}=\begin{bmatrix}1&1&1&1&1&1\\ \alpha_1&-\alpha_1&\alpha_2&-\alpha_2&-\dfrac{1}{\alpha_3}&\dfrac{1}{\alpha_3}\\ S_1&S_1&S_2&S_2&0&0\\ D_1&D_1&D_1&D_1&D_3&D_3\\ D_4&-D_4&D_5&-D_5&D_6&-D_6\\ D_7&-D_7&D_8&-D_8&0&0\end{bmatrix}\begin{bmatrix}A_1E_1\\ A_2E_2\\ A_3E_3\\ A_4E_4\\ A_5E_5\\ A_6E_6\end{bmatrix}E\tag{3.6}$$

where α_1^2,α_2^2 are roots of $\alpha^4+A\alpha^2+B=0$ and

$$\alpha_3^2=\frac{c^2}{c_T^2}-1,\qquad E_q=e^{i\xi\alpha_q z},\qquad E=e^{i\xi(x-ct)},\qquad q=1,2...6,$$

α_1^2, α_2^2 correspond to longitudinal and thermal waves whereas α_3^2 corresponds to the transverse wave which is not affected by the temperature variations

$$D_1=c_2\left(\frac{c^2}{c_T^2}-2\right),\qquad D_3=-2c_2,$$

$$D_4=2c_2\alpha_1,\qquad D_5=2c_2\alpha_2,\qquad D_6=\frac{c_2}{\alpha_3}\left(\frac{c^2}{c_T^2}-2\right),\tag{3.7}$$

$$D_7 = \alpha_1 S_1, \qquad D_8 = \alpha_2 S_2, \qquad c_T^2 = \frac{\mu}{\rho},$$

$$S_q = -\frac{\omega^* \tau^{\bullet} \zeta^2 \varepsilon_1}{\left(1+\alpha_q^2 - \tau\omega^* \zeta^2\right)}\left(1+\alpha_q^2\right), \qquad q = 1, 2, \tag{3.7}$$

$$\breve{\sigma}_{zz} = \frac{\sigma_{zz}}{i\xi}, \breve{\sigma}_{xz} = \frac{\sigma_{xz}}{i\xi}, \qquad \breve{T}' = \frac{T'}{i\xi}.$$

The matrix in (3.6) is the field matrix, describing the relationship between the wave amplitudes and the displacements, temperature, stresses and temperature gradient at any location in any layer. Its coefficients depend on the through-thickness position in the plate, the material properties of the layer, the frequency and the invariant plate and wave number. The origin of the coordinate may be placed arbitrary and may even be different for each layer because phase differences between layers can be accounted for by the phase of the complex wave number.

4. The transfer matrix method

The Transfer Matrix method works by condensing the layered system into a set of six equations relating the boundary conditions at the first interface to the boundary conditions at the last interface (Thomson, 1950). In the process, the equations for the intermediate interfaces are eliminated so that the fields in all of the layers of the plate are described solely in terms of the external boundary conditions.

The various parameters α_1, α_2, α_3, S_1, S_2, D_1, etc. are specialized to the material under consideration. Specializing Eq.(3.6) to the upper and bottom surface of each layer we can relate, after lengthy algebraic reductions and manipulations, the displacements, temperature, stresses and temperature gradient of the upper to those of the bottom as

$$P_m^+ = A_m P_m^-, \qquad m = 1, 2 \ldots N \tag{4.1}$$

where

$$P_m^{\pm} = \left\{\left[u, w, T, \breve{\sigma}_{zz}, \breve{\sigma}_{xz}, \breve{T}'\right]\right\}_m, \tag{4.2}$$

defines the variables column specialized to the upper and lower surfaces of the layer, m, respectively, and which provide us

$$A_m = X_m D_m X_m^{-1}$$

where is X_m is 6×6 square matrix and D_m is 6×6 diagonal matrix whose entries are $E_q = \exp\left(i\xi\alpha_q d\right)$ of Eq.(3.6). The matrix A_m constitutes the transfer matrix for the layer *m*. By applying the above procedure to each layer and invoking the continuity relations on the upper and bottom of each layer to those of its neighbors we can finally relate the displacements, temperature, stresses and temperature gradient at the top (top of layer (*N*)) of the plate to those at the bottom of the plate (bottom of layer (*1*)) via the transfer matrix multiplication. We can finally relate the displacements, stresses, temperature and temperature gradient at the top of the layered plate, $z = d$ to those at its bottom, $z = 0$ via the transfer matrix multiplication

$$P_m^+ = A_m P_m^- , \tag{4.3}$$

where $P_m^{\pm} = \{[u, w, T, \breve{\sigma}_{zz}, \breve{\sigma}_{xz}, \breve{T}']\}_m$ and the superscripts (+) and (-) designate quantities defined at the upper and bottom of the layer, respectively. By applying the above procedure to each layer and invoking the continuity relations on the upper and bottom of each layer to those of its neighbors we can finally relate the displacements, temperature, stresses and temperature gradient at the top (top of layer (n)) of the plate to those at the bottom of the plate (bottom of layer (1)) via the transfer matrix multiplication

$$\lfloor A_{ij} \rfloor = [A]_N [A]_{n-1} ......[A]_1 , \tag{4.4}$$

which leads to

$$\begin{bmatrix} u^{(N)} \\ w^{(N)} \\ T^{(N)} \\ \breve{\sigma}_{zz}^{(N)} \\ \breve{\sigma}_{xz}^{(N)} \\ \breve{T}'^{(N)} \end{bmatrix}_{z=d} = \begin{bmatrix} A_{11} & A_{12} & A_{13} & A_{14} & A_{15} & A_{16} \\ A_{21} & A_{22} & A_{23} & A_{24} & A_{25} & A_{26} \\ A_{31} & A_{32} & A_{33} & A_{34} & A_{35} & A_{36} \\ A_{41} & A_{42} & A_{43} & A_{44} & A_{45} & A_{16} \\ A_{51} & A_{52} & A_{53} & A_{54} & A_{55} & A_{16} \\ A_{61} & A_{62} & A_{63} & A_{64} & A_{65} & A_{66} \end{bmatrix} \begin{bmatrix} u^{(1)} \\ w^{(1)} \\ T^{(1)} \\ \breve{\sigma}_{zz}^{(1)} \\ \breve{\sigma}_{xz}^{(1)} \\ \breve{T}'^{(1)} \end{bmatrix}_{z=0} . \tag{4.5}$$

The characteristic equation for such a situation is obtained by invoking stress-free upper and bottom surfaces in Eq.(4.5).

Furthermore, for the traction-free boundary as well as thermally insulated conditions on the top and bottom surfaces of the plate, Eq.(4.6) can be rewritten as

$$\begin{bmatrix} u^{(N)} \\ w^{(N)} \\ T^{(N)} \\ 0 \\ 0 \\ 0 \end{bmatrix}_{z=d} = \begin{bmatrix} A_{11} & A_{12} & A_{13} & A_{14} & A_{15} & A_{16} \\ A_{21} & A_{22} & A_{23} & A_{24} & A_{25} & A_{26} \\ A_{31} & A_{32} & A_{33} & A_{34} & A_{35} & A_{36} \\ A_{41} & A_{42} & A_{43} & A_{44} & A_{45} & A_{46} \\ A_{51} & A_{52} & A_{53} & A_{54} & A_{55} & A_{56} \\ A_{61} & A_{62} & A_{63} & A_{64} & A_{65} & A_{66} \end{bmatrix} \begin{bmatrix} u^{(1)} \\ w^{(1)} \\ T^{(1)} \\ 0 \\ 0 \\ 0 \end{bmatrix}_{z=0} . \tag{4.7}$$

We obtain the characteristic equation as

$$\begin{vmatrix} A_{41} & A_{42} & A_{43} \\ A_{51} & A_{52} & A_{53} \\ A_{61} & A_{62} & A_{63} \end{vmatrix} = 0 . \tag{4.8}$$

Dispersion relation (4.8) is a transcendental function implicitly relating the phase velocity with the wave frequency. It is not possible to solve it analytically in an explicit form; hence numerical root searching tools have to be used to find phase velocity for every frequency point at a given wave number direction.

5. The global matrix method

Boundary conditions on top and bottom surfaces of the layered plate are:

$$\left(\sigma_{zz},\sigma_{xz},T'\right)\Big|_{z=\pm d/2}=0\,, \tag{5.1}$$

where d is the thickness of the plate.

Continuity conditions at the interface between the adjacent layers are given by

$$\boldsymbol{P}^{+}_{(n)}-\boldsymbol{P}^{-}_{(n+1)}=\boldsymbol{0}\,. \tag{5.2}$$

Equations (5.1) and (5.2) can be assembled into a matrix form as follows

$$\begin{bmatrix} \sum^{+}_{(n)} \\ \boldsymbol{P}^{-}_{(N)}-\boldsymbol{P}^{+}_{(N-1)} \\ \vdots \\ \boldsymbol{P}^{-}_{(n)}-\boldsymbol{P}^{+}_{(n-1)} \\ \vdots \\ \boldsymbol{P}^{-}_{(2)}-\boldsymbol{P}^{+}_{(1)} \\ \sum^{+}_{(1)} \end{bmatrix}=$$

$$=\begin{bmatrix} \left[F_{(N)}D^{+}_{(N)}\right]_{4-6} & & & & & \\ F_{(N)}D^{-}_{(N)} & -F_{(N-1)}D^{+}_{(N-1)} & & & & \\ & \mathbf{O} & \mathbf{O} & & & \\ & & F_{(n-1)}D^{-}_{(n-1)} & F_{(n)}D^{+}_{(n)} & & \\ & & & \mathbf{O} & \mathbf{O} & \\ & 0 & & & F_{(2)}D^{-}_{(2)} & -F_{(1)}D^{+}_{(1)} \\ & & & & & \left[F_{(1)}D^{-}_{(1)}\right]_{4-6} \end{bmatrix}\begin{bmatrix} A_{(N)} \\ A_{(N-1)} \\ \vdots \\ A_{(n)} \\ \vdots \\ A_{(2)} \\ A_{(1)} \end{bmatrix}=0 \tag{5.3}$$

where $\boldsymbol{D}^{+}_{(n)}=<e^{ikpz}>^{+}_{(n)}$, $\sum^{+}_{(n)}=\left(\breve{\sigma}_{zz},\breve{\sigma}_{xz},\breve{T}'\right)\Big|_{z=d/2}$ is the stress and thermal gradient components column on the top surface of the layered plate and $\sum^{(-)}_{(1)}=\left(\breve{\sigma}_{zz},\breve{\sigma}_{xz},\breve{T}'\right)\Big|_{z=d/2}$ is the stress thermal gradient components on the bottom surface, moreover the operator $[\ ]_{4-6}$ denotes extracting last three rows from the matrix in Eq.(3.6), thus both the first sub-matrix in the first row and the last sub-matrix in the last row are 3×6 matrix. For nontrivial solutions of Eq.(5.3), the determinant of the assembled matrix must vanish and consequently the dispersion relations for thermoelastic waves in a layered plate are

$$\begin{vmatrix} \left[F_{(N)}D^{+}_{(N)}\right]_{4-6} & & & & & & \\ F_{(N)}D^{-}_{(N)} & -F_{(N-1)}D^{+}_{(N-1)} & & & & & \\ & \mathbf{O} & \mathbf{O} & & & & \\ & & F_{(n-1)}D^{-}_{(n-1)} & F_{(n)}D^{+}_{(n)} & & & \\ & & & \mathbf{O} & \mathbf{O} & & \\ & 0 & & & F_{(2)}D^{-}_{(2)} & -F_{(1)}D^{+}_{(1)} & \\ & & & & & \left[F_{(1)}D^{-}_{(1)}\right]_{4-6} & \end{vmatrix} = 0\,. \qquad (5.4)$$

Although Eq.(5.4) is apparently different from Eq.(4.8) based on transfer matrix, both of them will yield the same dispersion results since they are deduced from the same traction-free surfaces and interface continuity conditions. Similarly to Eq.(4.8), when the material properties and fiber orientation of all layers are the same, Eq.(5.4) is expected to reduce to Eq.(14) of Verma (2001). Additionally, both symmetric modes and antisymmetric mode will be mixed together in the same manner as the transfer matrix method.

6. Thermoelastic symmetric and antisymmetric modes of separation

Although both the transfer matrix method and global matrix method can directly calculate a general thermoelastic layered plate with an arbitrary stacking sequence, it is unlikely to distinguish the symmetric and antisymmetric modes, even in symmetric multilayered plates, where the material properties of the multilayered plate are symmetric with respect to the center layer ($z = 0$ in the undeformed configuration), all solutions of the multilayered plate system decouple into the sum of a symmetric solution and an antisymmetric solution.

In order to overcome this shortcoming, a method is developed to decouple symmetric and antisymmetric modes in a symmetric laminate by imposing boundary conditions at the mid-plane of the plate. It is worth of noting that this method is only valid for symmetric laminates.

For elastic waves in a layered plate, boundary conditions on the top surfaces are

$$\left(\sigma_{zz}, \sigma_{xz}, T'\right)\Big|_{z=d/2} = 0\,, \qquad (6.1)$$

where d is the thickness of the laminate.

Continuity conditions at the interface between the adjacent layers are expressed in Eq.(5.2). Because of the symmetric structure of the laminate, one can consider only half of the laminate and then impose the symmetric or antisymmetric conditions on the stress and displacement components at the mid-plane. For nontrivial solutions of Eq.(6.1), the determinant of the matrix in Eq.(6.1) must be zero and consequently the dispersion relations for symmetric waves in a layered plate can be numerically obtained.

For symmetric modes, the boundary conditions at mid-plane can be expressed as

$$\left(w, T, \sigma_{xz}\right)\Big|_{z=0} = 0\,. \qquad (6.2)$$

Then these boundary conditions of the half-plate can be assembled into a matrix form

$$
\begin{bmatrix}
\Sigma^{+}_{(N)} \\
\boldsymbol{P}^{-}_{(N)}-\boldsymbol{P}^{+}_{(N-1)} \\
\vdots \\
\boldsymbol{P}^{-}_{(n)}-\boldsymbol{P}^{+}_{(n-1)} \\
\vdots \\
\boldsymbol{P}^{-}_{\left(\frac{N}{2}+1\right)}-\boldsymbol{P}^{+}_{\left(\frac{N}{2}+1\right)} \\
\Sigma^{-}_{\left(\frac{N}{2}\right)}
\end{bmatrix}
=
$$

$$
=\begin{bmatrix}
\left[F_{(N)}D^{+}_{(N)}\right]_{4-6} & & & & & & \\
F_{(N)}D^{-}_{(N)} & -F_{(N-1)}D^{+}_{(N-1)} & & & & & \\
 & \ddots & \ddots & & & & \\
 & & F_{(n+1)}D^{-}_{(n+1)} & F_{(n)}D^{+}_{(n)} & & & \\
 & & & \ddots & \ddots & & \\
 & 0 & & & F_{\left(\frac{N}{2}+1\right)}D^{-}_{\left(\frac{N}{2}+1\right)} & -F_{\left(\frac{N}{2}+1\right)}D^{+}_{\left(\frac{N}{2}+1\right)} & \\
 & & & & & \left[F_{\left(\frac{N}{2}\right)}D^{-}_{\left(\frac{N}{2}\right)}\right]_{2,3,5} &
\end{bmatrix}
\begin{bmatrix}
A_{(N)} \\
A_{(N-1)} \\
\vdots \\
A_{(n)} \\
\vdots \\
A_{\left(\frac{N}{2}+1\right)} \\
A_{\left(\frac{N}{2}\right)}
\end{bmatrix}
=0 \qquad (6.3)
$$

where $\boldsymbol{D}^{\pm}_{(n)}=<e^{ikpz}>^{\pm}_{(n)}$, $\sum^{+}_{(N)}=\left(\tilde{\sigma}_{zz},\tilde{\sigma}_{xz},\tilde{T}'\right)\Big|_{z=d/2}$, is the stress component column on the top surface of the laminated plate and $\sum^{-}_{\left(\frac{N}{2}\right)}=\left(w,T,\tilde{\sigma}_{xz}\right)^{t}\Big|_{z=0}$ is the stress components on the middle plane, moreover the operator $[\;]_{4-6}$ denotes extracting last four rows from the matrix in Eq.(3.6), similarly $[\;]_{2,3,5}$ indicates the second, third, fifth Eq.(3.6).

For nontrivial solutions of Eq.(6.4), the determinant of the matrix in Eq.(6.4), must be zero and consequently the dispersion relations for symmetric waves in a layered plate can be numerically obtained.

Similarly, the boundary conditions of antisymmetric modes on the mid-plane are

$$
\left(u,\sigma_{zz},T'\right)\Big|_{z=0}=0\,. \qquad (6.5)
$$

The assembled boundary condition matrix form of the half-plate has the following expression

$$
\begin{bmatrix}
\sum^{+}_{(N)} \\
\boldsymbol{P}^{-}_{(N)} - \boldsymbol{P}^{+}_{(N-1)} \\
\vdots \\
\boldsymbol{P}^{-}_{(n)} - \boldsymbol{P}^{+}_{(n-1)} \\
\vdots \\
\boldsymbol{P}^{-}_{\left(\frac{N}{2}+1\right)} - \boldsymbol{P}^{+}_{\left(\frac{N}{2}+1\right)} \\
\sum^{-}_{\left(\frac{N}{2}\right)}
\end{bmatrix} =
$$

$$
= \begin{bmatrix}
\left[F_{(N)}D^{+}_{(N)}\right]_{4-6} & & & & & & \\
F_{(N)}D^{-}_{(N)} & -F_{(N-1)}D^{+}_{(N-1)} & & & & & \\
 & \ddots & \ddots & & & & \\
 & & F_{(n+1)}D^{-}_{(n+1)} & F_{(n)}D^{+}_{(n)} & & & \\
 & & & \ddots & \ddots & & \\
 & 0 & & & F_{\left(\frac{N}{2}+1\right)}D^{-}_{\left(\frac{N}{2}+1\right)} & -F_{\left(\frac{N}{2}+1\right)}D^{+}_{\left(\frac{N}{2}+1\right)} & \\
 & & & & & \left[F_{\left(\frac{N}{2}\right)}D^{-}_{\left(\frac{N}{2}\right)}\right]_{1,4,6} &
\end{bmatrix}
\begin{bmatrix}
A_{(N)} \\ A_{(N-1)} \\ \vdots \\ A_{(n)} \\ \vdots \\ A_{\left(\frac{N}{2}+1\right)} \\ A_{\left(\frac{N}{2}\right)}
\end{bmatrix} = 0 \tag{6.6}
$$

where $\sum^{-}_{\left(\frac{N}{2}\right)} = \left(u, \sigma_z, T'\right)^{tr}\Big|_{z=0}$ are the displacements stress and temperature gradient components on the middle plane, operator $[\ ]_{1,4,6}$ denotes extracting the first, fourth and sixth rows from the matrix in Eq.(3.6).

To guarantee nontrivial solutions of Eq.(6.6), the determinant of the matrix in Eq.(6.6) should be zero.

7. Special cases

7.1. Smooth interface

If a smooth contact interface is introduced within the plate at an arbitrary location, say the interface between layer (m) and $(m+1)$, then the previous analysis must be modified. In this sense, we may now consider the plate to be composed of two subplates, the top subplate with *n-m* layers, where $1 \le m \le n$. The appropriate conditions for the smooth contact surface are

$$
\begin{aligned}
& w^{(m+1)} = w^{(m)}, \qquad T^{(m+1)} = T^{(m)}, \\
& \sigma_{zz}^{(m+1)} = \sigma_{zz}^{(m)}, \qquad T'^{(m+1)} = T'^{(m)}, \\
& \sigma_{xz}^{(m+1)} = \sigma_{xz}^{(m)} = 0 \qquad \text{at} \qquad z^{(m+1)} = 0, \qquad z^{(m)} = d^{(m)}.
\end{aligned} \tag{7.1}
$$

First, we construct the top subplate macro transfer matrix by truncating A_{ij} starting from the top as

$$\left[T_{ij}\right] = \left[a_{ip}\right]_n \left[a_{pq}\right]_{n-1} .. \left[a_{lj}\right]_{m+1} . \tag{7.2}$$

Next, we construct the bottom subplate macro-transfer matrix from the remaining part of A_{ij}

$$\left[H_{ij}\right] = \left[a_{ip}\right]_m \left[a_{pq}\right]_{m-1} ... \left[a_{lj}\right]_1 , \tag{7.3}$$

here $\left[A_{ij}\right] = \left[T_{ip}\right]\left[H_{pj}\right]$. Invoking the smooth conditions (7.1) leads to a modified matrix $\left[A_{ij}\right]$ relating the field variables at the top and bottom of the plate and hence replacing Eq.(4.5).

7.2. Displacements, temperature, stresses and temperature gradient within the plate

The displacements, temperature, stresses and temperature gradient within an arbitrary layer, l, at an arbitrary point, say $z^{(l)} = z_0$, are obtained by exactly the same procedure as we use to obtain the relationship between the plate boundary responses, given by Eq.(4.5). That is the matrix transfer technique is used to extend the boundary solution at $z = 0$ to locations within the plate

$$\begin{bmatrix} u^{(n)} \\ w^{(n)} \\ T^{(n)} \\ \overline{\sigma}_{zz}{}^{(n)} \\ \overline{\sigma}_{xz}{}^{(n)} \\ \overline{T}'^{(n)} \end{bmatrix}_{z=z_0} = \begin{bmatrix} F_{11} & F_{12} & F_{13} & F_{14} & F_{15} & F_{16} \\ F_{21} & F_{22} & F_{23} & F_{24} & F_{25} & F_{26} \\ F_{31} & F_{32} & F_{33} & F_{34} & F_{35} & F_{36} \\ F_{41} & F_{42} & F_{43} & F_{44} & F_{45} & F_{16} \\ F_{51} & F_{52} & F_{53} & F_{54} & F_{55} & F_{16} \\ F_{61} & F_{62} & F_{63} & F_{64} & F_{65} & F_{66} \end{bmatrix}_m \begin{bmatrix} u^{(1)} \\ w^{(1)} \\ T^{(1)} \\ \overline{\sigma}_{zz}{}^{(1)} \\ \overline{\sigma}_{xz}{}^{(1)} \\ \overline{T}'^{(1)} \end{bmatrix} \tag{7.4}$$

where, for the plate with all rigidly bonded interfaces

$$\left[F_{ij}\right] = \left[b_{ip}\right]_l \left[a_{pq}\right]_{l-1} ... \left[a_{rj}\right]_1 , \qquad 1 \le l \le N \tag{7.5}$$

and the entries of $\left[b_{ij}\right]$ in (7.5) are obtained from the micro-transfer matrix for layer l $\left(\left[a_{ij}\right]\right)$ by replacing z in Eq.(7.2) by z_0. Thus, $\left[F_{ij}\right]$ in Eq.(7.5) is a transfer matrix that relates the displacements, temperature, stresses and temperature gradient at $z^{(l)} = z_0$ with those at $z = 0$.

If a smooth interface is present between layers m and $m+1$, the displacements, temperature, stresses and temperature gradient are again given by Eqs (7.4) and (7.5) with the constraint $1 \le l \le N$ and $\left[A_{ij}\right]$ in (4.5) replaced by the appropriate micro-transfer matrix. On the other hand, the displacements, temperature, stresses and temperature gradient above the smooth interface, upon invoking the continuity and zero shear stress conditions in Eq.(7.1), are given by replacing $\left[F_{ij}\right]$ in Eq.(7.5) by

$$[F] = [G]_{6\times 6} [Q]_{6\times 4} [R]_{4\times 6} \tag{7.6}$$

where

$$[G]=\begin{bmatrix} G_{11} & G_{12} & G_{13} & G_{14} & G_{15} & G_{16} \\ G_{21} & G_{22} & G_{23} & G_{24} & G_{25} & G_{26} \\ G_{31} & G_{32} & G_{33} & G_{34} & G_{35} & G_{36} \\ G_{41} & G_{42} & G_{43} & G_{44} & G_{45} & G_{16} \\ G_{51} & G_{52} & G_{53} & G_{54} & G_{55} & G_{16} \\ G_{61} & G_{62} & G_{63} & G_{64} & G_{65} & G_{66} \end{bmatrix}, \qquad [Q]=\begin{bmatrix} -\dfrac{T_{42}}{T_{41}} & -\dfrac{T_{43}}{T_{41}} & -\dfrac{T_{44}}{T_{41}} & -\dfrac{T_{45}}{T_{41}} \\ 1 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 \\ 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 1 \\ 0 & 0 & 0 & 0 \end{bmatrix},$$

$$[R]=\begin{bmatrix} B_{21} & B_{22} & B_{23} & B_{24} & B_{25} & B_{26} \\ B_{31} & B_{32} & B_{33} & B_{34} & B_{35} & B_{36} \\ B_{41} & B_{42} & B_{43} & B_{44} & B_{45} & B_{46} \\ B_{51} & B_{52} & B_{53} & B_{54} & B_{55} & B_{56} \end{bmatrix}$$

and
$$\left[G_{ip}\right] = \left[b_{iq}\right]_l \left[a_{qj}\right]_{l-1} \ldots \left[a_{sp}\right]_{m+1}, \qquad 1 \le l \le N \tag{7.7}$$

Equation (7.4) together with Eqs (7.5) and (7.6) give the displacements, temperature, stresses and temperature gradient at arbitrary points within a plate in terms of the lower boundary response for the case of all rigidly bonded interfaces and the case of a smooth interface between layers (m) and $(m+1)$, respectively.

7.3. Classical case

This case corresponds to the situation when the strain and temperature fields are not coupled with each other. In this case the thermo-mechanical coupling constant is identically zero.

7.4. Zero coupling

When the coupling constant $\varepsilon_1 = 0$, the thermal wave which is influenced by the thermal relaxation time τ_0 gets decoupled from its counterpart in elasticity (Nayfeh and Taylor, 1990).

7.5. Coupled thermoelasticity

When the thermal relaxation time $\tau_0 \to 0$, then the results obtained in the analysis reduce to conventional coupled theory of thermoelasticity.

8. Conclusion and discussion

The development of a general purpose model for thermoelastic wave propagation for heat conducting isotropic layered plates is shown. The obtained model can account for thermo and visco-elastic heat conducting isotropic materials, single or multi-layered structures, and free or leaky systems. Displacements, temperature, thermal stresses and temperature gradient within a multilayered plate in generalized theories of thermoelasticity are also obtained, using the matrix transfer technique. Rigidly bonded and smooth interface conditions are also considered as a special case to stimulate de-bonding of two materials layers and the model developed will be of value in material characterization and providing other quantitative information on thermo-mechanical, strength related properties of advanced materials. The Global Matrix methodis power ful and the same matrix may be used for all types of solutions. The disadvantage is that the global matrix may be large and therefore the solution may be relatively slow. The method can also be used to calculate dispersion relations for a one layered plate case, when the layer number $N = 1$, or the material properties of all layers are the same.

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Nomenclature

A_m	– transfer matrix constitutes
c	– phase velocity
C_e	– specific heat at constant deformation
d	– thickness of the plate
D_m	– 6×6 diagonal matrix
$[G],[Q],[R]$	– denotes matrices
i	– imaginary number
K	– thermal conductivity
N	– number of layers
$P_m^{\pm}$	– column specialized to the upper and lower surfaces of the layer, *m*
T'	– temperature gradient
$\breve{T}'$	– temperature gradient
$\boldsymbol{u}$	– isplacement vector
$\dot{\boldsymbol{u}}$	– denotes time differentiation
$u_{,xx}, w_{,xz}$	– comma notation is used for spatial derivatives
U_1, U_2, U_3	– constant amplitudes
x	– propagation direction
X_m	– 6×6 square matrix
z	– normal to the interfaces
α_t	– coefficient of thermal expansion
ε_1	– coupling constant
γ	– thermoelastic coupling parameter
λ, μ	– Lame' parameters
θ_0	– uniform reference temperature
θ, T	– temperature change above
ρ	– mass density
$\sigma_{zz}, \sigma_{zz}, \sigma_{xz}$	– thermal stresses
$\tau_1, \tau_0, \tau_2, \tau^{\bullet}$	– thermal relaxation times
ω	– circular frequency
ω^*	– characteristic frequency
ξ	– wave number

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