Brief note

SINGULARITY BEHAVIOR OF FLOW IN A CURVED PIPE

M.A.H. KHAN*
Department of Mathematics
Bangladesh University of Engineering & Technology
Dhaka-1000, BANGLADESH
e-mail: mahkhan@math.buet.ac.bd

This work is dedicated to my advisor Professor Philip G. Drazin (1934-2002) Unicewrsity of Bristol, U.K.

Singularity behavior of a flow in a curved pipe is analyzed. A series related to a similarity parameter is obtained by using algebraic programming language MAPLE. The series is then analyzed by various generalizations of the approximate methods. The result is not conclusive, but we show approximately the dominating behavior of the flow near the (unphysical) singularity point.

Key words: approximate method, complex bifurcation.

1. Introduction

The first theoretical analysis of flow in a curved pipe was carried out by Dean (1927; 1928). He compared his results with the experimental data obtained by Eustice in 1911, and found a good agreement. In essence, Dean's solution shows that the streamlines exhibit helical motion when the curvature of the pipe is small; this motion divides into two "*screw*" motions on both sides of the central plane of the pipe. In his first paper, Dean could not describe how the curvature of the pipe affects the flow near the boundary. Later, in his second paper, he remarked that the curvature reduces the pressure gradient on the boundary, so that the flux ratio decreases there.

Following Dean, many mathematicians such as White (1929), Taylor (1929), Itö (1959), Mori and Nakayama (1965) worked on the problem. A power series for Dean's solution was analysed by Topakoglu (1967), Larrain and Bonilla (1970) and Van Dyke (1978). Van Dyke (1978) used a series for the flux ratio, while Dean's conclusions were based on the first two terms of the series for the friction ratio. Other contributions have made use of boundary layer theory.

The remainder of this paper is as follows: We describe the mathematical model of the flow in a curved rotating pipe in 2. In 3, we use approximate methods to analyse the dominant singularity behavior of the solution.

2. Formulation of the problem

We consider the steady motion of an incompressible viscous fluid through a circular curved pipe of uniform cross section. We follow Dean (1928) in using the spherical coordinates (r, θ, φ). Let Oz be the axis perpendicular to the central line through the pipe so that θ is the angle between the radius vector and the Oz axis. See Fig.1.

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* To whom correspondence should be addressed

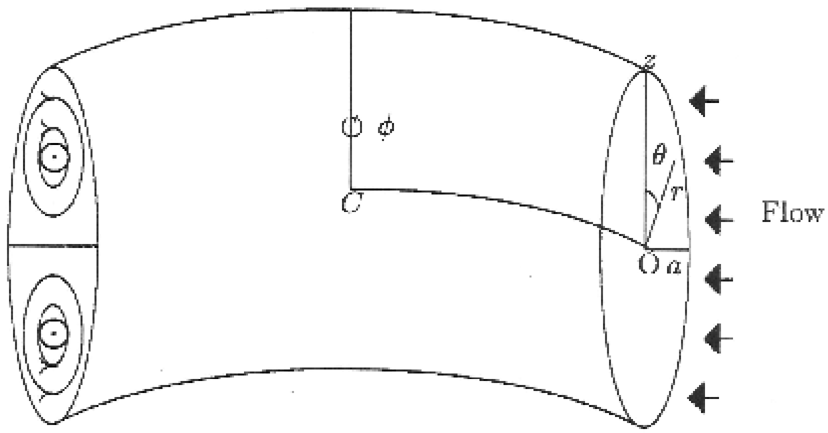

Fig.1. A schematic diagram of the flow in a curved pipe.

C is the centre of the section of the pipe by a plane that makes an angle φ with a fixed axial plane. Denote by *u*, *v* and *w* the components of the velocity vector defined as follows: consider a cross section of the pipe as in Fig.1. Then *u* is the component in the radial direction in the plane of the cross section, *v* is the component in the tangential direction in the same plane, and *w* is the component along the central line through the pipe. The radius of the cross section is denoted by *a* and the radius of the circle around which the pipe is coiled is denoted by *L*. The general direction of flow is assumed to be that of φ increasing.

Then the Navier-Stokes equations can be written as

$$u\frac{\partial u}{\partial r}+\frac{v}{r}\frac{\partial u}{\partial \theta}-\frac{w^2\sin\theta}{L+r\sin\theta}-\frac{v^2}{r}=-\frac{\partial}{\partial r}\left(\frac{p}{\rho}\right)-\nu\left[\left(\frac{1}{r}\frac{\partial}{\partial\theta}+\frac{\cos\theta}{L+r\sin\theta}\right)\left(\frac{\partial v}{\partial r}+\frac{v}{r}-\frac{1}{r}\frac{\partial u}{\partial\theta}\right)\right], \tag{2.1}$$

$$u\frac{\partial v}{\partial r}+\frac{v}{r}\frac{\partial v}{\partial \theta}-\frac{w^2\cos\theta}{L+r\sin\theta}+\frac{uv}{r}=-\frac{1}{r}\frac{\partial}{\partial \theta}\left(\frac{p}{\rho}\right)+\nu\left[\left(\frac{\partial}{\partial r}+\frac{\sin\theta}{L+r\sin\theta}\right)\left(\frac{\partial v}{\partial r}+\frac{v}{r}-\frac{1}{r}\frac{\partial u}{\partial\theta}\right)\right], \tag{2.2}$$

$$\begin{gathered}u\frac{\partial w}{\partial r}+\frac{v}{r}\frac{\partial w}{\partial \theta}+\frac{uw\sin\theta}{L+r\sin\theta}+\frac{vw\cos\theta}{L+r\sin\theta}=-\frac{1}{L+r\sin\theta}\frac{\partial}{\partial\phi}\left(\frac{p}{\rho}\right)+\\+v\left[\left(\frac{\partial}{\partial r}+\frac{1}{r}\right)\left(\frac{\partial w}{\partial r}+\frac{w\sin\theta}{L+r\sin\theta}\right)+\frac{1}{r}\frac{\partial}{\partial\theta}\left(\frac{1}{r}\frac{\partial w}{\partial\theta}+\frac{w\cos\theta}{L+r\sin\theta}\right)\right],\end{gathered} \tag{2.3}$$

with the continuity equation

$$\frac{\partial u}{\partial r}+\frac{u}{r}+\frac{u\sin\theta}{L+r\sin\theta}+\frac{1}{r}\frac{\partial v}{\partial\theta}+\frac{v\cos\theta}{L+r\sin\theta}=0\,. \tag{2.4}$$

We assume that *L* is large, so that a/L is small. We shall in fact consider the limit $a/L\to 0$. We shall also assume that the motion is the same in each cross section of the pipe, that is

$$-\frac{\partial}{L\partial\phi}\left(\frac{p}{\rho}\right)=\frac{G}{\rho} \tag{2.5}$$

where G is the pressure gradient along the central line. Let $f(r,\theta)$ be the stream function so that we have

$$ru=-\frac{\partial f}{\partial\theta}, \qquad v=\frac{\partial f}{\partial r}.$$

We introduce the dimensionless variables

$$\tilde{\psi}=\frac{f}{\nu}, \qquad \tilde{w}=\frac{w}{w_0}, \qquad \tilde{r}=\frac{r}{a} \tag{2.6}$$

where w_0 is the maximum speed in a straight pipe under the same axial pressure gradient. Then by eliminating the pressure gradient terms, we can write Eqs (2.1)-(2.3) in the following form (dropping the tildes for simplicity)

$$\nabla^2 w+4=\frac{1}{r}\left(\frac{\partial\psi}{\partial r}\frac{\partial w}{\partial\theta}-\frac{\partial\psi}{\partial\theta}\frac{\partial w}{\partial r}\right), \tag{2.7}$$

$$\nabla^4\psi=\overline{K}w\left(-\cos\theta\frac{\partial w}{\partial r}+\frac{\sin\theta}{r}\frac{\partial w}{\partial\theta}\right)+\frac{1}{r}\left(-\frac{\partial\psi}{\partial\theta}\frac{\partial}{\partial r}+\frac{\partial\psi}{\partial r}\frac{\partial}{\partial\theta}\right)\nabla^2\psi \tag{2.8}$$

where

$$\nabla^2=\frac{\partial^2}{\partial r^2}+\frac{1}{r}\frac{\partial}{\partial r}+\frac{1}{r^2}\frac{\partial^2}{\partial\theta^2},$$

$$\overline{K}=2\left(\frac{w_0 a}{\nu}\right)^2\frac{a}{L} \quad \text{and} \quad G=\frac{4\mu w_0}{a^2}, \tag{2.9}$$

with the boundary conditions

$$w=1, \quad \psi=0 \quad \text{at} \quad r=0 \quad \text{and} \quad w=\psi=\frac{\partial\psi}{\partial r}=0 \quad \text{at} \quad r=1. \tag{2.10}$$

Now, Eqs (2.7), (2.8) with the boundary condition (2.10) can be solved easily when $\overline{K}=0$. The solution then corresponds to a Poiseuille flow for a straight circular pipe. In order to investigate the flow for a curved pipe, it is therefore natural to expand the solution in powers of the parameter $\overline{K}$

$$w=w_0+\overline{K}w_1+\overline{K}^2 w_2+\ldots \quad \text{and} \quad \psi=\overline{K}\psi_1+\overline{K}^2\psi_2+\ldots.$$

By using MAPLE, we find the first 25 coefficients of the series for the axial velocity w and the stream function ψ in terms of the coordinates r and θ. The coefficients are computed symbolically and are thus free of round-off error. The first two coefficients of the series for the axial velocity w are

$$w_0 = 1 - r^2,$$
$$w_1 = \left(\frac{19}{23040} r - \frac{1}{576} r^3 + \frac{1}{768} r^5 - \frac{1}{2304} r^7 + \frac{1}{23040} r^9 \right) \sin\theta. \tag{2.11}$$

3. Result and discussion

For the analysis, we choose the series for the functional

$$\frac{dw}{dr}(r = 1, \quad \theta = \pi/2),$$

which is related to the tangential shear stress at the wall of the pipe. From the results shown in Tabs 1 and 2, it appears that there is a pair of complex conjugate singularities along the imaginary axis in the complex $\overline{K}$ plane. These singularities have the form

$$\frac{dw}{dr}(1, \pi/2) \sim \overline{A} + \overline{B}\left(\overline{K} \pm \overline{K}_c\right)^{\overline{\alpha}} \quad \text{as} \quad \overline{K} \to \pm \overline{K}_c, \tag{3.1}$$

with, $\overline{\alpha} = 1/2$. These findings are comparable to those of Van Dyke (1978). We have plotted an approximate bifurcation diagram in Fig.2. Van Dyke (1978) used an Euler transformation to show that a solution exists for large $\overline{K}$. As can be seen in Fig.3, the method of Drazin and Tourigny (1996) succeeds in continuing the real solution beyond the circle of convergence of the series. However, to learn something about the properties of the flow, we need to consider yet larger values of $\overline{K}$. But we do not have enough series coefficients to do this accurately.

Table 1. Estimates of $\overline{K}_c$ and $\overline{\alpha}$ in Eq.(3.1) by high-order differential approximants Khan (2002; 2003).

D	N	$\overline{K}_c, N$	$\overline{\alpha}\ N$
2	7	- 14.85549 ± 602.09315i	0.376181981± .2153940 i
3	13	- 0.2254334 ± 587.00711i	0.454255624 - 0 .0080953 i
4	18	0.0018806 ± 585.77978i	0.500724160 - 0.0001890 I

Table 2. Estimates of $\overline{K}_c$, $\overline{A}$ and $\overline{B}$ in Eq.(3.1) by the Drazin-Tourigny (1996) method.

d	N	$\overline{K}_c, N$	$\overline{A}_N$	$\overline{B}_N$
2	5	- 0.115356 ± 0.623030 i	0.441070 - 0.2382118i	0.113566 +0.136524i
3	9	0.82090911± 588.474987 i	-1.869690 - 0.506063i	0.005503 +0.023646i
4	14	0.16830677 ± 585.694255i	-1.871674 - 0.496850i	0.004964 +0.022470i
5	20	- 0.00038021±585.788540i	-1.871356 - 0.497227i	0.004962 +0.022582i

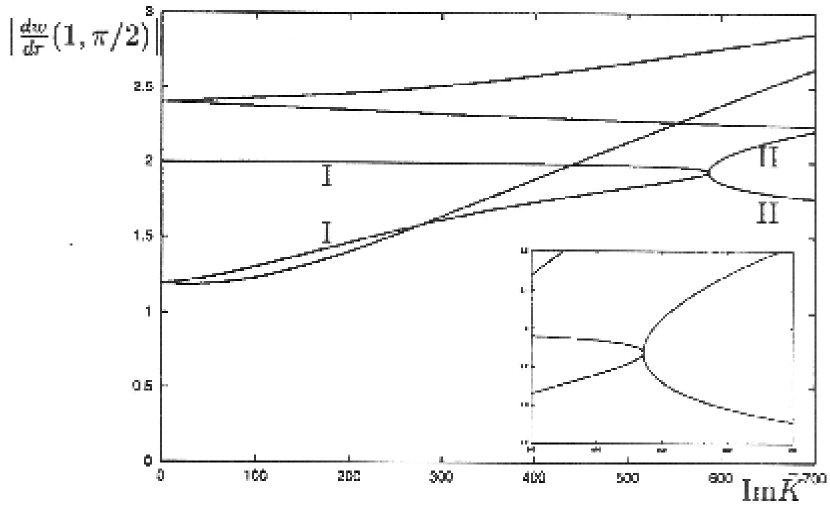

Fig.2. Approximate bifurcation diagram (curves I and II) in the $\left(\mathrm{Im}\overline{K}, \left|\frac{dw}{dr}(1, \pi/2)\right|\right)$ plane obtained by the Drazin-Tourigny (1996) method for $d = 5$. The inset in the lower right-hand box is a zoom-in on the bifurcation point. Note that the other curves are spurious.

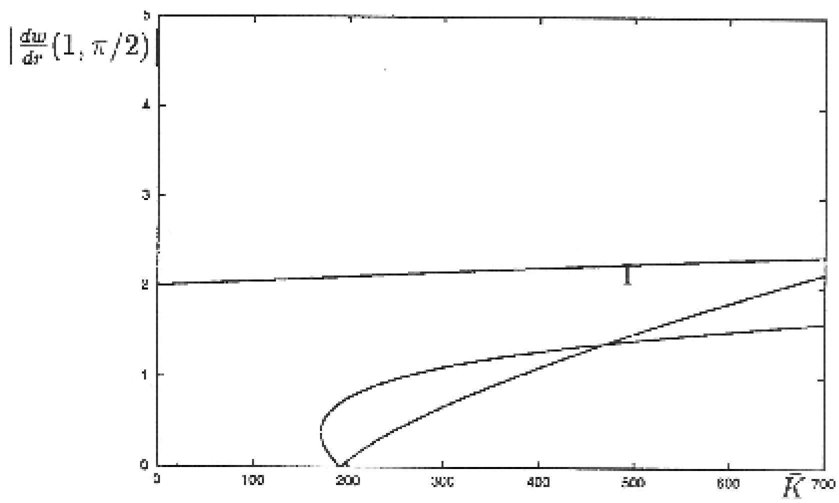

Fig.3. Approximate solution diagram (curve I) for $\overline{K} > 0$ obtained by the Drazin-Tourigny (1996) method for $d = 5$. The other curves are spurious.

4. Conclusion

We have verified that the convergence of the series Eq.(2.11) is limited by a pair of singularities located along the imaginary axis in the complex $\overline{K}$ plane. Again, the dominant singularities are complex, as previously noted by Van Dyke. Such complex singularities have no physical interpretation, and thus tell us very little about the flow. Dean described the secondary motion in the flow that occurs as the relevant expansion parameter increases. In principle, we can learn something about the aspects of the flow by using our (Drazin-Tourigny, 1996 and Khan, 2002) powerful approximate methods to continue the series solution well beyond the nearest (unphysical) singularities. In practice, however, we simply do not have enough series coefficients at our disposal to exploit the approximate methods. This unfortunate situation arises all too often in applications to problems of fluid dynamics.

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