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WAVES IN THERMOELASTIC MATERIALS WITH MEMORY IN GENERALIZED THERMOELASTICITY

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Many studies on classical thermoelastoicity have been denoted to materials with memory Nunziato, Chen and Gurtin, whereas few on generalized thermoelasticity address materials with memory. The present paper deals with the wave propagation in materials with memory in generalized thermoelasticity. Plane progressive waves and Rayleigh waves have been discussed in detail. The results show appreciable differences with those in the usual classical thermoelasticity theory.

Key words: materials with memory, generalized thermoelasticity, plane progressive waves, Rayleigh waves.

1. Introduction

In the mechanics of continuous media, a material is said to have hereditary characteristics or memory if its behavior at time t is specified by the past experience of the body up to time t. The theory of heat conduction in materials with memory has drawn the attention of many researchers. The motivation was provided by an unpleasant feature of the classical linear theory of heat conduction viz., that a thermal disturbance produced at some point in the body felt instantaneously at all other points. This contradicts the relativistic principle that energy cannot be propagated at speeds exceeding the velocity of light. Gurtin and Pipkin (1968) first established a general non-linear theory of heat conduction in rigid materials with memory for which thermal disturbances propagate with finite speed. They assumed that the response functional viz., the entropy, free energy and heat-flux depend on the present value of the temperature and the integrated histories of the temperature and the temperature gradient.

Nunziato (1971) considered a slightly different memory theory of heat-conduction. He assumed the response functional to depend on the histories upto the present time of the temperature and the temperature gradient. In his theory, heat conduction depends also on the present value of the temperature gradient so that Fourier's law of heat conduction is obtained as a particular case, if $k(0)$, the instantaneous conductivity, is non-zero. On the other hand, if $k(0)=0$, Nunziato's heat conduction equation agrees with that of Gurtin and Pipkin. Chen and Gurtin (1970) extended the theory presented by Gurtin and Pipkin to deformable media. They started with the constitutive assumptions that the response functional viz., the stress, entropy, free energy and heat-flux depend on the present values of the temperature and the deformation gradient and the integrated histories of the deformation gradient, temperature and the temperature gradient. McCarthy (1970), on the other hand, assuming the response functionals to be dependent on the present values of the

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temperature and the deformation gradient, histories of the deformation gradient and the temperature and the integrated history of the temperature gradient, developed a theory of thermo-mechanical materials with memory.

Ghosh (1972), however, assumed the response functionals to be dependent on the present values of the temperature and the deformation gradient and the histories of the temperature, deformation gradient and the temperature gradient. His assumptions are thus less restrictive than those of Chen and Gurtin and of McCarthy. He extended Nunziato's theory of heat conduction to deformable bodies and deduced the generalized stress-strain relation and the coupled equation of heat conduction for materials with memory and obtained the linearised forms of these equations. The stress-strain relation is similar to that for a linear thermo-visco-elastic solid given by Eringen (see Eringen (1967), p.367-368). The heat conduction equation for a thermo-visco-elastic solid is the same as that for the thermo-elastic solid (Eringen, 1967). It is, thus, observed that the stress-strain relation and the heat conduction equation obtained by Ghosh (1972) is more general than those of others in the sense that the equations of classical thermoelasticity as well as those of classical thermo-visco-elasticity can all be obtained as particular cases. The stress-strain relation and the coupled equation of heat conduction in the linear coupled thermo-elasticity theory given by Chadwick (1960) for thermoelastic materials (see Müller (1967)) can be obtained as particular cases from those deduced by Ghosh. This is to be expected, since for "thermoelastic" materials the response functionals depend only on the present values of the temperature, temperature gradient, and the deformation gradient and so "thermoelastic" materials are special cases of "thermoelastic materials with memory".

Chen, Amos, Nunziato and McCarthy have studied propagation of thermoelastic waves. Chen (1969) studied the amplitude variation of the temperature rate waves of arbitrary form, assuming that the constitutive relations of the material are given by the linearised theory derived by Gurtin and Pipkin (1968). Amos and Chen (1970) obtained the speed of propagation of thermal waves in a one dimensional case, while Nunziato (1971) determined the attenuation and the speed of plane temperature and temperature rate waves. McCarthy (1970) studied the propagation of "first order" waves using the equations derived by him McCarthy (1970).

The recent works in this line by Bhattacharyya and Kumar (1991), Matinez and Quintanilla (1995), Chirita and Quintanilla (1997), Iesan and Quintanilla (2002), Zhou *et al*. (2003) may be of worth mentioning.

In the present paper, we study plane progressive thermoelastic waves and Rayleigh waves in generalized thermoelasticity in a general form using the more general equations deduced by Ghosh (1972). It is observed that as in the usual coupled thermoelasticity theory, there exist two types of propagating plane waves, one is the quasi-elastic wave and the other is the quasi-thermal wave (Chadwick, 1960, p.283). Both the waves, however, exhibit dispersion and attenuation. The modified elastic wave speed approaches the "instantaneous" value of the classical compressional wave velocity at high frequencies and its "equilibrium" value modified by the coupling factor at low frequencies. Its attenuation coefficient tends to a constant value at large frequencies and is proportional to the square of the frequency at low frequencies. Similar features are observed in the usual coupled thermoelasticity theory. But a new feature exhibited by plane propagating purely elastic waves in materials with memory is that these waves are subject to both dispersion and attenuation. The purely elastic wave speed approaches the "instantaneous" value of the classical compressional wave velocity at high frequencies and its equilibrium value at low frequencies. The attenuation coefficient is proportional to the square of the frequency at low frequencies and tends to a constant value at high frequencies.

For thermal waves both the velocity and the attenuation coefficient are proportional to the square root of the frequency at low and high frequencies.

The dispersion equation for Rayleigh waves has also been obtained. It has a form analogous to that in the usual linear coupled thermoelasticity theory.

2. Formulation of equation

Following the analysis of Nunziato (1971) regarding the deduction of stress-strain relations in classical thermoelasticity and Dhaliwal and Sing (1980), Lord and Shulman (1967) and Green and Lindsay

(1972) in generalized thermoelasticity, the stress-strain-temperature relations and heat conduction equations are written below:

i) assuming isotropy and linearity, the equation of heat conduction or energy equation can be written as

$$\alpha(0)\left(\dot{\theta}+t_0\ddot{\theta}\right)+\int_0^\infty \alpha'(s)\left[\dot{\theta}(\boldsymbol{x},t-s)+t_0\ddot{\theta}(\boldsymbol{x},t-s)\right]ds+\eta(0)\mathrm{div}\left[\dot{\boldsymbol{u}}(\boldsymbol{x},t)+\delta_{lk}t_0\ddot{\boldsymbol{u}}(\boldsymbol{x},t)\right]+$$
$$\int_0^\infty \eta'(s)\mathrm{div}\left[\dot{\boldsymbol{u}}(\boldsymbol{x},t-s)+\delta_{lk}t_0\ddot{\boldsymbol{u}}(\boldsymbol{x},t-s)\right]ds=k(0)\nabla^2\theta(\boldsymbol{x},t)+\int_0^\infty k'(s)\nabla^2\theta(\boldsymbol{x},t-s)ds. \tag{2.1}$$

ii) Stress-strain-temperature relations are

$$\sigma_{ij}(\boldsymbol{x},t)=\left[\lambda(0)e(\boldsymbol{x},t)-\beta(0)\left\{\theta(\boldsymbol{x},t)+t_1\dot{\theta}(\boldsymbol{x},t)\right\}\delta_{ij}+2\mu(0)e_{ij}(\boldsymbol{x},t)\right]+$$
$$+\int_0^\infty\left[\left\{\lambda'(s)e(\boldsymbol{x},t-s)-\beta'(s)\left\{\theta(\boldsymbol{x},t-s)+t_1\dot{\theta}(\boldsymbol{x},t-s)\right\}\delta_{ij}\right\}+2\mu'(s)e_{ij}(\boldsymbol{x},t-s)\right]ds. \tag{2.2}$$

iii) Equations of motion are

$$\sigma_{ij,j}(\boldsymbol{x},t)=\rho\ddot{\boldsymbol{u}}(\boldsymbol{x},t). \tag{2.3}$$

iv) Strain-displacement relations

$$e_{ij}=\frac{1}{2}\left(u_{i,j}+u_{j,i}\right),\qquad i,\ j=1,2,3\,. \tag{2.4}$$

In Eqs (2.1) and (2.2) $\alpha(s), k(s)$ are the energy-temperature relaxation function and the heat-conduction relaxation function respectively, which in the case of “thermoelastic” materials correspond to specific heat and conductivity respectively. $\lambda(s), \mu(s)$ may be called the Lamé relaxation functions corresponding to the usual Lamé constants. $\beta(s), \eta(s)$ are the relaxation functions corresponding to β, η where β is the coefficient of the temperature deviation in the stress-strain relation, and η is the coupling constant. θ denotes temperature deviation, $\boldsymbol{u}(\boldsymbol{x}, t)$ is the displacement vector.

It should be noted that the relations

$$\beta(0)=\left[3\lambda(0)+2\mu(0)\right]\alpha_T\,,\qquad \eta(0)=T_0\beta(0)$$

where α_T is the coefficient of linear expansion and T_0 is the reference temperature, hold though they are not valid for other values of the argument s (they are valid for “thermoelastic” materials when $\alpha(s), \beta(s), \lambda(s), k(s), \eta(s), \mu(s)$ are all constants). If the relaxation functions $\alpha(s), \beta(s), \lambda(s)\ k(s), \eta(s)$, $\mu(s)$ are constants Eqs (2.1) and (2.2) reduce to the corresponding equations in the usual linear thermoelasticity theory.

In the absence of body forces the equation of motion in deformable materials with memory becomes, by the use of Eq.(2.2)

$$\mu(0)\nabla^2\boldsymbol{u}(\boldsymbol{x},t)+\{\lambda(0)+\mu(0)\}\text{grad div}\boldsymbol{u}(\boldsymbol{x},t)-\beta(0)\left[\text{grad}\theta(\boldsymbol{x},t)+t_1\text{grad}\dot{\theta}(\boldsymbol{x},t)\right]+$$
$$+\int_0^\infty[\mu'(s)\nabla^2\boldsymbol{u}(\boldsymbol{x},t-s)+\{\lambda'(s)+\mu'(s)\}\text{grad div}\boldsymbol{u}(\boldsymbol{x},t-s)+ \tag{2.5}$$
$$-\beta'(s)\left[\text{grad}\theta(\boldsymbol{x},t-s)+t_1\text{grad}\dot{\theta}(\boldsymbol{x},t-s)\right]ds=\rho\ddot{\boldsymbol{u}}(\boldsymbol{x},t).$$

Taking the divergence of Eq.(2.5), we obtain

$$\{\lambda(0)+2\mu(0)\}\nabla^2 e(\boldsymbol{x},t)-\beta(0)\left[\nabla^2\theta(\boldsymbol{x},t)+t_1\frac{\partial}{\partial t}\nabla^2\theta(\boldsymbol{x},t)\right]+$$
$$+\int_0^\infty\left[\{\lambda'(s)+2\mu'(s)\}\nabla^2 e(\boldsymbol{x},t-s)-\beta'(s)\left\{\nabla^2\theta(\boldsymbol{x},t-s)+t_1\frac{\partial}{\partial t}\nabla^2\theta(\boldsymbol{x},t-s)\right\}\right]ds=\rho\ddot{e}(\boldsymbol{x},t) \tag{2.6}$$

where $e(\boldsymbol{x},t)=\text{div}\boldsymbol{u}(\boldsymbol{x},t)$.

3. Solution of the problem

Let us assume the following form of the plane wave solutions of Eqs (2.1) and (2.6)

$$[e(\boldsymbol{x},t),\theta(\boldsymbol{x},t)]=(e_0,\theta_0)\exp[i\omega(t-\boldsymbol{x}\hat{n}/c)] \tag{3.1}$$

where $\omega>0$ and $\hat{n}$ is a unit vector.

Substituting Eq.(3.1) in Eqs (2.6) and (2.1), we get

$$\left[\{\lambda(0)+2\mu(0)\}+\{\bar{\lambda}'(\omega)+2\bar{\mu}'(\omega)\}-\rho c^2\right]e_0-\left[\beta(0)(1+t_1 i\omega)+\bar{\beta}'(\omega)\{1+t_1 i\omega\}\right]\theta_0=0, \tag{3.2a}$$

$$\left[\alpha(0)\left(i\omega-t_0\omega^2\right)+\bar{\alpha}'(\omega)\left\{i\omega-t_0\omega^2\right\}\right]\theta_0+\left[(i\omega)+\delta_{lk}t_0\left(-\omega^2\right)\right]\eta(0)e_0+$$
$$+e_0\left[i\omega+t_0\delta_{lk}\left(-\omega^2\right)\right]\bar{\eta}'(\omega)=\left[k(0)+\bar{k}'(\omega)\right]\left(-\omega^2/c^2\right)\theta_0,$$

or,

$$\left[\left(i\omega-t_0\omega^2\right)\{\gamma(0)+\bar{\gamma}'(\omega)\}+\left(\omega^2/c^2\right)\{k(0)+\bar{k}'(\omega)\}\right]\theta_0+$$
$$+\left(i\omega-\delta_{lk}t_0\omega^2\right)\{\eta(0)+\bar{\eta}'(\omega)\}e_0=0 \tag{3.2b}$$

where

$$\left(\bar{\beta}'(\omega),\bar{\gamma}'(\omega),\bar{\lambda}'(\omega),\bar{\mu}'(\omega),\bar{\eta}'(\omega),\bar{k}'(\omega)\right)=$$
$$=\int_0^\infty(\beta'(s),\gamma'(s),\lambda'(s),\mu'(s),\eta'(s),k'(s))\exp(-i\omega s)ds. \tag{3.3}$$

Putting

$$\alpha(0)+\bar{\alpha}'(\omega)=A,\quad \beta(0)+\bar{\beta}'(\omega)=B,\quad \lambda(0)+2\mu(0)+\bar{\lambda}'(\omega)+2\bar{\mu}'(\omega)=L,$$
$$k(0)+\bar{k}'(\omega)=K,\quad \mu(0)+\bar{\mu}'(\omega)=M,\quad \eta(0)+\bar{\eta}'(\omega)=N, \tag{3.4}$$

and eliminating e_0, θ_0 between Eqs (3.2a) and (3.2b), we get

$$(1+i\omega t_0\delta_{lk})\rho c^4 A - c^2[(1+i\omega\delta_{lk}t_0)AL + \rho i\omega K + (1+t_1 i\omega)(1+\delta_{lk}t_0 i\omega)BN] + i\omega LK = 0 . \quad (3.5)$$

If $t_0 = t_1 = 0$, we get back the results (6a) and (6b) of Chakraborty (1976) in classical thermoelasticity.

The roots $C^2_{1,2}$ of Eq.(3.5) are given by

$$C^2_{1,2} = [AL(1+i\omega t_0\delta_{lk}) + \rho i\omega K + (1+i\omega t_1)(1+i\omega t_0\delta_{lk})BN \pm D]/[2\rho A(1+i\omega t_0\delta_{lk})] \quad (3.6)$$

where

$$D^2 = \{AL(1+i\omega t_0\delta_{lk}) + \rho i\omega K + (1+i\omega t_1)(1+i\omega t_0\delta_{lk})BN\}^2 - 4i\omega LK(1+i\omega t_0\delta_{lk})\rho A . \quad (3.7)$$

If $\eta(s) \neq 0$, the root C_1 corresponds to a quasi-elastic wave and the root C_2 corresponds to a quasi-thermal wave, because for $\eta(s) = 0$ and $t_0 = 0$

$$C_1 = (L/\rho)^{1/2} , \qquad C_2 = (i\omega K/A)^{1/2} ,$$

and these roots correspond, in the uncoupled problem, to a pure elastic and thermal wave respectively.

Let us now analyse the roots of the dispersion Eq.(3.5) for low and high frequencies.

(a) Low frequency

Expanding $\alpha'(\omega)$ for small ω in the form

$$\alpha'(\omega) = \alpha'(0) + \left[\frac{d}{d\omega}\overline{\alpha}'(\omega)\right]_{\omega=0} \omega + ... , \quad (3.8)$$

we can write

$$\overline{\alpha}'(\omega) \approx \alpha(\infty) - \alpha(0) - i\alpha_1\omega$$

where

$$\alpha_1 = \int_0^\infty s\alpha'(s)ds . \quad (3.9)$$

If we assume the existence of integrals of the type

$$\alpha_n = \int_0^\infty s^n \alpha^{(n)}(s)ds , \qquad n > 1 ,$$

we can retain the terms of higher order in ω in Eq.(3.8).

Thus, for a small ω, we can write

$$A \approx \alpha(\infty) - i\alpha_1\omega, \qquad B \approx \beta(\infty) - i\omega\beta_1, \qquad M \approx \mu(\infty) - i\mu_1\omega,$$
$$L \approx \lambda(\infty) + 2\mu(\infty) - i\omega(\lambda_1 + 2\mu_1) = l(\infty) - i\omega l_1, \tag{3.10}$$
$$K \approx k(\infty) - i\omega k_1, \qquad N \approx \eta(\infty) - i\omega\eta_1$$

where $\alpha_1, \beta_1, \lambda_1, \mu_1, \eta_1, k_1$ are integrals, assumed to converge, of the type (3.9). Using the results Eq.(3.10) in Eq.(3.6) and extracting the square root, we get for a small ω

$$C_1 \approx \left[\frac{l(\infty)}{\rho}\right]^{1/2} \times \left[1 + \frac{\beta(\infty)\eta(\infty)}{l(\infty)\alpha(\infty)}\right]^{1/2} \times$$
$$\times\left[1 + i\frac{\omega}{2}\left\{\begin{array}{l}\dfrac{t_0\delta_{lk}l(\infty)\alpha(\infty) - \alpha_1 l(\infty) - l_1\alpha(\infty) - \beta_1\eta(\infty) - \eta_1\beta(\infty) + (t_1 + t_0\delta_{lk})\beta(\infty)\eta(\infty)}{l(\infty)\alpha(\infty) + \beta(\infty)\eta(\infty)} \\ -\dfrac{t_0\alpha(\infty)\delta_{lk} - \alpha_1}{\alpha(\infty)} + \rho\dfrac{k(\infty)\beta(\infty)\eta(\infty)}{(l(\infty)\alpha(\infty) + \beta(\infty)\eta(\infty))^2}\end{array}\right\}\right], \tag{3.11}$$

$$C_2 \approx \left[\frac{\omega k(\infty)}{2l(\infty)}\right]^{1/2} \times \left[1 + \frac{\beta(\infty)\eta(\infty)}{l(\infty)\alpha(\infty)}\right]^{-1/2} \times$$
$$\times\left[\left\{1 + \frac{\omega}{2l(\infty)}(t_0\alpha(\infty)\delta_{lk} - \alpha_1)\right\} + i\left\{1 - \frac{\omega}{2l(\infty)}(t_0\alpha(\infty)\delta_{lk} - \alpha_1)\right\}\right]. \tag{3.12}$$

If V_1, V_2 denote the wave velocities and ν_1, ν_2 the attenuation coefficients for the modified elastic and thermal waves respectively, then

$$V_i = \frac{|C_i|^2}{\operatorname{Re} C_i}, \tag{3.13}$$

$$\nu_i = \omega\frac{\operatorname{Im} C_i}{|C_i|^2}, \qquad i = 1, 2. \tag{3.14}$$

Form Eqs (3.11)-(3.14) we obtain

$$V_1 \approx \left[\frac{\lambda(\infty) + 2\mu(\infty)}{\rho}\right]^{1/2}\left[1 + \frac{\beta(\infty)\eta(\infty)}{l(\infty)\alpha(\infty)}\right]^{1/2}\left[1 + 0(\omega^2)\right], \tag{3.15}$$

$$\nu_1 \approx -\frac{\omega^2}{2}\left[\frac{\rho}{\lambda(\infty)+2\mu(\infty)}\right]^{1/2}\left[1+\frac{\beta(\infty)\eta(\infty)}{l(\infty)\alpha(\infty)}\right]^{-1/2}\times$$
$$\times\left[\begin{array}{l}\dfrac{\alpha_1 l(\infty)+l_1\alpha(\infty)+\beta_1\eta(\infty)+\eta_1\beta(\infty)-t_0\delta_{lk}l(\infty)\alpha(\infty)-(t_1+t_0\delta_{lk})\beta(\infty)\eta(\infty)}{l(\infty)\alpha(\infty)+\eta(\infty)\beta(\infty)}+\\ -\dfrac{l_1-\alpha(\infty)\delta_{lk}t_0}{\alpha(\infty)}-\rho\dfrac{k(\infty)\beta(\infty)\eta(\infty)}{(l(\infty)\alpha(\infty)+\beta(\infty)\eta(\infty))^2}\end{array}\right], \quad (3.16)$$

$$V_2 \approx \left[\frac{2\omega k(\infty)}{l(\infty)}\right]^{1/2}\left[1+\frac{\beta(\infty)\eta(\infty)}{l(\infty)\alpha(\infty)}\right]^{-1/2}\left[1-\frac{\omega}{2}\left\{\frac{\alpha(\infty)t_0\delta_{lk}-\alpha_1}{l(\infty)}\right\}\right], \quad (3.17)$$

$$\nu_2 \approx \left[\frac{\omega l(\infty)}{2k(\infty)}\right]^{1/2}\left[1+\frac{\beta(\infty)\eta(\infty)}{l(\infty)\alpha(\infty)}\right]^{1/2}\left[1-\frac{\omega}{2}\left\{\frac{\alpha(\infty)t_0\delta_{lk}-\alpha_1}{l(\infty)}\right\}\right]. \quad (3.18)$$

(b) High frequency

Integrating each of the transforms in Eq.(3.3) by parts, we obtain

$$\bar{\alpha}'(\omega) = -\frac{\alpha'(0)}{\omega}i - \frac{\alpha''(0)}{\omega^2} - \frac{\bar{\alpha}'''(\omega)}{\omega^2}, \quad (3.19)$$

$$\bar{\beta}'(\omega) = -\frac{\beta'(0)}{\omega}i - \frac{\beta''(0)}{\omega^2} - \frac{\bar{\beta}'''(\omega)}{\omega^2} \qquad \text{etc.} \quad (3.20)$$

where

$$\bar{\alpha}'''(\omega) = \int_0^\infty \alpha'''(s)\exp(-i\omega s)ds \qquad \text{etc.}$$

Hence for a large ω, we can write

$$A \approx \alpha(0) - \frac{i}{\omega}\alpha'(0), \qquad B \approx \beta(0) - \frac{i}{\omega}\beta'(0),$$

$$K \approx k(0) - \frac{i}{\omega}k'(0), \qquad L \approx (\lambda(0)+2\mu(0)) - \frac{i}{\omega}(\lambda'(0)+2\mu'(0)), \quad (3.21)$$

$$M \approx \mu(0) - \frac{i}{\omega}\mu'(0), \qquad M \approx \eta(0) - \frac{i}{\omega}\eta'(0).$$

The substitution of Eq.(3.21) in Eq.(3.6) yields, for a large ω

$$C_1 \approx \left[\frac{\omega k(0)}{2\alpha(0)}\right]^{1/2}\left[1+\frac{1}{2\omega}\left\{\frac{\beta(0)\eta(0)}{\rho k(0)}+\frac{k'(0)}{k(0)}-\frac{\alpha'(0)}{\alpha(0)}+\frac{1}{\rho k(0)}(t_1+t_0\delta_{lk})\beta(0)\eta(0)\right\}+\right.$$
$$\left.+i\left\{1-\frac{1}{2\omega}\left(\frac{\beta(0)\eta(0)}{\rho k(0)}+\frac{k'(0)}{k(0)}-\frac{\alpha'(0)}{\alpha(0)}+\frac{1}{\rho k(0)}(t_1+t_0\delta_{lk})\beta(0)\eta(0)\right)\right\}\right], \tag{3.22}$$

$$C_2 \approx \left[\frac{\lambda(0)+2\mu(0)}{\rho}\right]^{1/2}\left[1-\frac{i}{2\omega}.\frac{\lambda'(0)+2\mu'(0)}{\lambda(0)+2\mu(0)}\right]. \tag{3.23}$$

The wave velocities and attenuation coefficients for the elastic and thermal waves are given by

$$V_1 \approx \left[\frac{2\omega k(0)}{\alpha(0)}\right]^{1/2}\left[1-\frac{1}{2\omega}\left\{\frac{\beta(0)\eta(0)}{\rho k(0)}+\frac{k'(0)}{k(0)}-\frac{\alpha'(0)}{\alpha(0)}+\frac{1}{\rho k(0)}(t_1+t_0\delta_{lk})\beta(0)\eta(0)\right\}\right], \tag{3.24}$$

$$v_1 \approx \left[\frac{\omega\alpha(0)}{2k(0)}\right]^{1/2}\left[1-\frac{1}{2\omega}\left\{\frac{\beta(0)\eta(0)}{\rho k(0)}+\frac{k'(0)}{k(0)}-\frac{\alpha'(0)}{\alpha(0)}+\frac{1}{\rho k(0)}(t_1+t_0\delta_{lk})\beta(0)\eta(0)\right\}\right], \tag{3.25}$$

$$V_2 \approx \left[\frac{\lambda(0)+2\mu(0)}{\rho}\right]^{1/2}\left[1+0\left(\frac{1}{\omega^2}\right)\right], \tag{3.26}$$

$$v_2 \approx -\frac{1}{2}\left(\frac{\rho}{\lambda(0)+2\mu(0)}\right)^{1/2}\left[\frac{\lambda'(0)+2\mu'(0)}{\lambda(0)+2\mu(0)}\right]\left[1+0\left(\frac{1}{\omega^2}\right)\right]. \tag{3.27}$$

It is observed that both the modified elastic and thermal waves in a generalized thermoelastic medium exhibit damping and dispersion. The modified thermal elastic wave speed, unlike that in the classical result approaches the instantaneous value of the classical compressional wave velocity at high frequencies and its equilibrium value is modified due to the presence of the coupling factor at low frequencies. Both the quasi-thermal wave speed and attenuation coefficient are proportional to the square root of the frequency at low frequency. Results for ‘thermoelastic’ materials may be obtained from the results of Eqs (3.15)-(3.18) and (3.24)-(3.27) by taking the relaxation functions to constants. They agree with those given by Chadwik (1960) and Nowacki (1962).

We neglect the coupling between the temperature and the displacement fields by putting

$$\eta(s)=\eta(0)=\eta(\infty)=0\,,$$

in Eqs (3.15)-(3.18) and (3.24)-(3.27) and get the corresponding results for the purely elastic and the purely thermal modes. Results for the purely thermal mode agree with those given by Nunziato (1971).

For the purely elastic mode

$$V_E \approx \left[\frac{\lambda(\infty)+2\mu(\infty)}{\rho}\right]^{1/2}\left[1+\frac{\omega^2}{4}\left(\frac{\lambda_1+2\mu_1}{\lambda(\infty)+2\mu(\infty)}\right)^2\right] \quad \text{as} \quad \omega\to 0\,, \tag{3.28}$$

$$v_E \approx -\frac{\omega^2}{2}\left[\frac{\rho}{\lambda(\infty)+2\mu(\infty)}\right]^{1/2}\left[\frac{\lambda_1+2\mu_1}{(\lambda(\infty)+2\mu(\infty))}\right] \qquad \text{as} \qquad \omega \to 0 \tag{3.29}$$

where V_E, ν_E are the wave speed and attenuation coefficient for the purely elastic waves; and

$$V_E \approx \left[\frac{\lambda(0)+2\mu(0)}{\rho}\right]^{1/2}\left[1+\frac{1}{4}\omega^2\left(\frac{\lambda_1+2\mu_1}{\lambda(0)+2\mu(0)}\right)^2\right] \qquad \text{as} \qquad \omega \to \infty, \tag{3.30}$$

$$v_E \approx -\frac{1}{2}\left(\frac{\rho}{\lambda(0)+2\mu(0)}\right)^{1/2}\left[\frac{\lambda'(0)+2\mu'(0)}{\lambda(0)+2\mu(0)}\right]\left[1+0\left(\frac{1}{\omega^2}\right)\right] \qquad \text{as} \qquad \omega \to \infty. \tag{3.31}$$

The results of Eqs (3.28)-(3.31) show that the purely elastic wave has its modification, the quasi-elastic wave is subject to damping and dispersion. This is a feature exhibited by the elastic waves only in material with memory both in the cases of classical and generalized thermoelasticity. The attenuation coefficients at low and high frequencies are proportional to $\lambda_1+2\mu_1$ and $\lambda'(0)+2\mu'(0)$ respectively. These vanish if $\lambda(s)+2\mu(s)$ is a constant i.e., in classical elasticity. Thus attenuation of a pure elastic wave occurs in materials with memory. However, it may be pointed out that the effect of generalized thermoelasticity is found to be absent. The velocity for pure elastic waves tends to different limits, in general, according as $\omega \to 0$ or $\omega \to \infty$; but if $\lambda(s)+2\mu(s)$ is a constant, as in the classical elasticity V_E becomes independent of ω and assumes the constant value of the classical compressional wave velocity.

4. Rayleigh waves

Putting the displacement vector $\boldsymbol{u}$ in the form

$$\boldsymbol{u} = \text{grad}\phi + \text{rot}\,\boldsymbol{A}, \qquad \boldsymbol{A} = (0, 0, \psi),$$

as the sum of irrotational and solenoidal components in Eqs (2.1) and (2.5), we get

$$\begin{aligned}&\left[\{\lambda(0)+2\mu(0)\}\nabla^2 - \rho\frac{\partial^2}{\partial t^2}\right]\phi(\boldsymbol{x},t) - \beta(0)\left[\theta(\boldsymbol{x},t)+t_1\dot{\theta}(\boldsymbol{x},t)\right]+\\ &+\int_0^\infty\left[\{\lambda'(s)+2\mu'(s)\}\nabla^2\phi(\boldsymbol{x},t-s) - \beta'(s)\{\theta(\boldsymbol{x},t)+t_1\dot{\theta}(\boldsymbol{x},t)\}\right]ds = 0,\end{aligned} \tag{4.1}$$

$$\mu(0)\nabla^2\psi(\boldsymbol{x},t) - \rho\ddot{\psi}(\boldsymbol{x},t) + \int_0^\infty \mu'(s)\nabla^2\psi(\boldsymbol{x},t-s)ds = 0, \tag{4.2}$$

$$\alpha(0)\left(\dot{\theta}+t_0\ddot{\theta}\right)+\int_0^\infty \gamma'(s)\left[\dot{\theta}(\boldsymbol{x},t-s)+t_0\ddot{\theta}(\boldsymbol{x},t-s)\right]ds+\eta(0)\left\{\nabla^2\dot{\phi}(\boldsymbol{x},t)+\delta_{lk}t_0\nabla^2\ddot{\phi}(\boldsymbol{x},t)\right\}$$
$$\int_0^\infty \eta'(s)\left\{\nabla^2\dot{\phi}(\boldsymbol{x},t-s)+\delta_{lk}t_0\ddot{\phi}(\boldsymbol{x},t-s)\right\}ds=k(0)\nabla^2\theta(\boldsymbol{x},t)+\int_0^\infty k'(s)\nabla^2\theta(\boldsymbol{x},t-s)ds\,. \tag{4.3}$$

Using the substitution (see Nowacki, 1962)

$$\left\{\theta(\boldsymbol{x},t),\phi(\boldsymbol{x},t),\psi(\boldsymbol{x},t)\right\}=\left\{\hat{\theta}(x_1),\hat{\phi}(x_1),\hat{\psi}(x_1)\right\}\exp\left[i\omega\left(t-x_2/C\right)\right],$$

and Eq.(3.4) in Eqs (4.1)-(4.3) we get

$$\Big[KL\left(D^2-\omega^2/C^2\right)^2-i\omega\left(D^2-\omega^2/C^2\right)\left\{(1+it_0\omega)AL+(1+i\omega t_1)(1+it_0\omega)BN+i\rho\omega K\right\}+$$
$$-i\omega^3\rho(1+it_0\omega)A\Big]\left(\hat{\theta},\hat{\phi}\right)=0\,, \tag{4.4}$$

$$\left[M\left(D^2-\omega^2/C^2\right)+\rho\omega^2\right]\hat{\psi}=0 \tag{4.5}$$

where $\qquad D\equiv\dfrac{d}{dx_1}\,.$

For surface waves we assume the solutions of Eqs (4.4)-(4.5) in the form

$$\hat{\psi}(x_1)=A_3\exp(-\gamma x_1)\,, \tag{4.6}$$

and

$$\hat{\phi}(x_1)=A_1\exp(-x_1\chi_1)+A_2\exp(-x_1\chi_2) \tag{4.7}$$

where χ_1,χ_2 are the roots of the equation

$$\Big[KL\left(\chi^2-a^2\right)^2-i\omega\left(\chi^2-a^2\right)\left\{(1+it_0\omega)AL+(1+i\omega t_1)(1+i\omega t_0)BN+i\rho\omega K\right\}\Big]+$$
$$-i\omega^3\rho(1+t_0 i\omega)A=0\,. \tag{4.8}$$

If t_0 and t_1 equal to zero, Eq.(4.8) totally coincides with that obtained by Chakraborti (1976). Now putting the expression (4.7) in Eq.(4.1), we get

$$\hat{\theta}_1(x_1)=\frac{1}{B(1+i\omega t_1)}\Big[\Big\{\left(L\left(\chi_1^2-a^2\right)+\rho\omega^2\right)A_1\exp(-\chi_1x_1)+$$
$$+\left(L\left(\chi_2^2-a^2\right)+\rho\omega^2\right)A_2\exp(-\chi_2x_1)\Big\}\Big]. \tag{4.9}$$

The stress components $\sigma_{11}(\boldsymbol{x},t),\sigma_{12}(\boldsymbol{x},t)$ given by Eq.(2.2) can be written in terms of the potential functions ϕ,ψ as

$$\sigma_{12} = 2\mu(0)\phi_{,12}(\boldsymbol{x},t) + 2\int_0^\infty \mu'(s)\phi_{,12}(\boldsymbol{x},t-s)ds + \\ + \mu(0)\left[\psi_{,22}(\boldsymbol{x},t) - \psi_{,11}(\boldsymbol{x},t)\right] + \int_0^\infty \mu'(s)\left[\psi_{,22}(\boldsymbol{x},t-s) - \psi_{,11}(\boldsymbol{x},t-s)\right]ds, \tag{4.10}$$

$$\sigma_{11}(\boldsymbol{x},t) = \rho\ddot{\phi}(\boldsymbol{x},t) + 2\mu(0)\left[\psi_{,12}(\boldsymbol{x},t) - \phi_{,22}(\boldsymbol{x},t)\right] + \\ + 2\int_0^\infty \mu'(s)\left[\psi_{,12}(\boldsymbol{x},t-s) - \phi_{,22}(\boldsymbol{x},t-s)\right]ds. \tag{4.11}$$

Thus the boundary conditions

$$\frac{\partial\theta(0,x_2;t)}{\partial x_1} + h\theta(0,x_2;t) = 0, \qquad \sigma_{11}(0,x_2;t) = 0 = \sigma_{12}(0,x_2;t),$$

will yield the system of equations

$$\left(2a^2 - \tau^2\right)A_1 + \left(2a^2 - \tau^2\right)A_2 + 2ia\gamma A_3 = 0, \\ 2ia\chi_1 A_1 + 2ia\chi_2 A_2 - \left(2a^2 - \tau^2\right)A_3 = 0, \\ (h - \chi_1)n_1 A_1 + (h - \chi_2)n_2 A_2 = 0 \tag{4.12}$$

where

$$n_{1,2} = \chi_{1,2}^2 + \sigma^2 - a^2, \quad \sigma^2 = \rho\omega^2/L, \quad a^2 = \omega^2/C^2, \quad \tau = \rho\omega^2/M, \quad \gamma = \left(a^2 - \tau^2\right)^{1/2}.$$

The condition of consistency of the system of Eq.(4.12) gives

$$\left(2a^2 - \tau^2\right)^2 = 4a^2\left(a^2 - \tau^2\right)^{1/2} \frac{(\chi_1 n_2 - \chi_2 n_1)h + \chi_1\chi_2(n_1 - n_2)}{(n_1 - n_2)h + (\chi_1 n_1 - \chi_2 n_2)},$$

i.e.,

$$\left(2a^2 - \tau^2\right)^2\left[\left(\sigma^2 - a^2\right) + \chi_1^2 + \chi_2^2 + \chi_1\chi_2\right] - 4a^2\left(a^2 - \tau^2\right)^{1/2}\chi_1\chi_2(\chi_1 + \chi_2) = \\ = h\left[\left(2a^2 - \tau^2\right)^2(\chi_1 + \chi_2) + 4a^2\left(a^2 - \tau^2\right)^{1/2}\left\{\left(\sigma^2 - a^2\right) - \chi_1\chi_2\right\}\right]. \tag{4.13}$$

For small ω, $\omega << 1$, this Eq.(4.13) becomes

$$\left[i+i^{1/2}\omega^{1/2}\left\{\frac{1}{C^2}-\frac{\rho\alpha(\infty)}{l(\infty)\alpha(\infty)+\beta(\infty)\eta(\infty)}\right\}^{1/2}\left\{\frac{l(\infty)\alpha(\infty)+\beta(\infty)\eta(\infty)}{l(\infty)k(\infty)}\right\}^{-1/2}\right]\times$$

$$\times\left[\left(\frac{2}{C^2}-\frac{\rho}{\mu(\infty)}\right)^2-\frac{4}{C^2}\left(\frac{1}{C^2}-\frac{\rho}{\mu(\infty)}\right)^{1/2}\left\{\frac{1}{C^2}-\frac{\rho\mu(\infty)}{l(\infty)\alpha(\infty)+\beta(\infty)\eta(\infty)}\right\}^{1/2}\right]=$$

$$=\frac{h}{\omega^{1/2}}\left\{\left(\frac{2}{C^2}-\frac{\rho}{\mu(\infty)}\right)^2\left[i^{1/2}+\omega^{1/2}\left\{\frac{l(\infty)\alpha(\infty)+\beta(\infty)\eta(\infty)}{\alpha(\infty)k(\infty)}\right\}^{-1/2}\times\right.\right.$$

$$\left.\times\left\{\frac{1}{C^2}-\frac{\rho\alpha(\infty)}{l(\infty)\alpha(\infty)+\beta(\infty)\eta(\infty)}\right\}^{1/2}\right]+$$

$$\left.-\frac{4}{C^2}\left(\frac{1}{C^2}-\frac{\rho}{\mu(\infty)}\right)^{1/2}i^{1/2}\left[\frac{1}{C^2}-\frac{\rho\alpha(\infty)}{l(\infty)\alpha(\infty)+\beta(\infty)\eta(\infty)}\right]^{1/2}\right\}\times$$

$$\times\left\{\frac{l(\infty)\alpha(\infty)+\beta(\infty)\eta(\infty)}{l(\infty)k(\infty)}\right\}^{-1/2}\left[1+0\left(\omega^{1/2}\right)\right]. \tag{4.14}$$

The result (4.14) shows that, on neglecting terms of order $\omega^{1/2}$, the velocity C ceases to depend upon the frequency ω and the thermal constant h (Chadwick, 1960).

We then have

$$\left[2-\frac{\rho C^2}{\mu(\infty)}\right]^2=4\left[1-\frac{\rho C^2}{\mu(\infty)}\right]^{1/2}\left[1-\frac{\rho C^2\alpha(\infty)}{\alpha(\infty)l(\infty)+\beta(\infty)\eta(\infty)}\right]^{1/2}.$$

This result is analogous to the corresponding result in classical thermoelasticity.

5. Numerical results and discussion

Numerical results are obtained for $k=1$, $l=2$, $\beta=0.089$, $\eta=18.07$, $\alpha=0.01$, $\alpha_1=0.002$ and different values of ω. The wave velocity V_2 and the attenuation coefficient ν_2 are calculated for CDT and GLT for low frequency and the corresponding graphs are drawn to show the relationship between wave velocity and frequency and that between attenuation coefficient and frequency.

It is seen from the graphs that velocity and attenuation increase due to the increase of frequency in both cases of CDT and GLT. They show slightly non-linearity in behaviour but are different in magnitude.

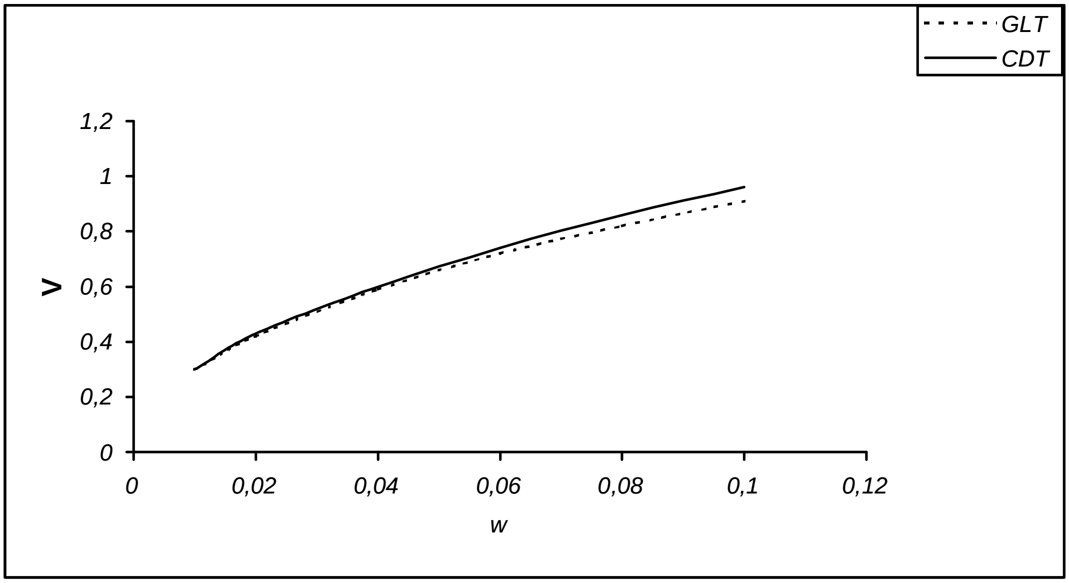

Fig.1. Wave velocity V_2 for different values of ω (low frequency) in GLT and CDT.

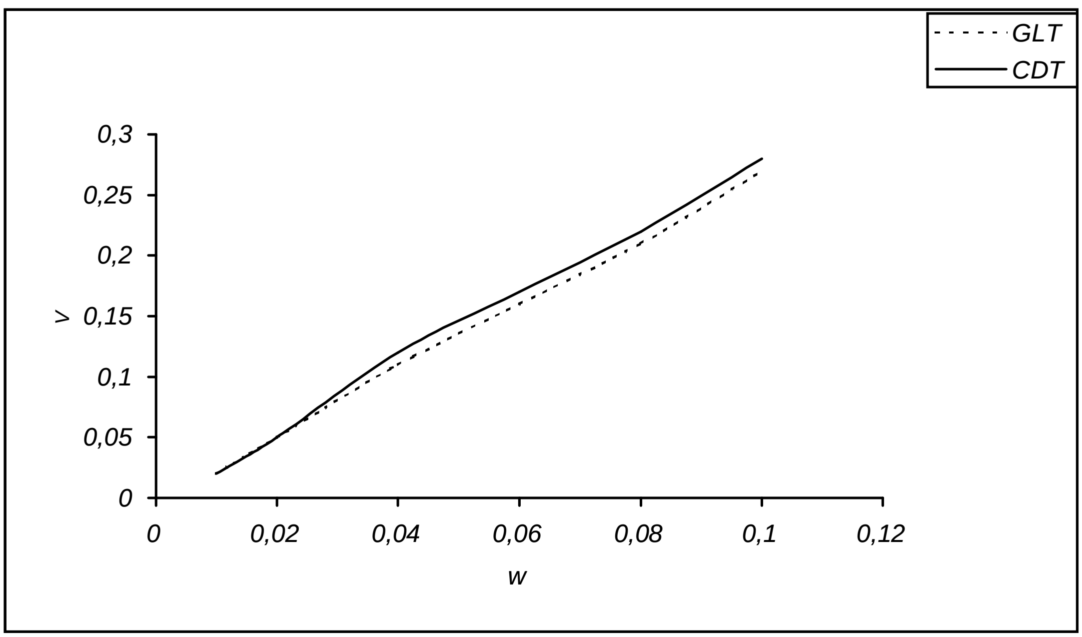

Fig.2. Attenuation coefficient v_2 for different values of ω (low frequency) GLT and CDT.

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Nomenclature

$k(s)$ – heat conduction relaxation function
T_0 – reference temperature
$\boldsymbol{u}(\boldsymbol{x}, t)$ – displacement vector
$\alpha(s)$ – energy temperature relaxation function
α_T – coefficient of linear expansion
β – coefficient of the temperature deviation in the stress-strain relation
$\beta(s)$ – elaxation functions corresponding to β
η – coupling constant
$\eta(s)$ – relaxation functions corresponding to η
$\lambda(s)$ – Lamé relaxation functions corresponding to the usual Lamé constants
$\mu(s)$ – Lamé relaxation functions corresponding to the usual Lamé constants
θ – temperature deviation

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