# MODELING MICROPOLAR ELECTRORHEOLOGICAL FLUIDS

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In general, *electrorheological fluids* are suspensions consisting of solid particles and a carrier oil. If such a suspension is exposed to an electric field, the particles form structures which have essentially the direction of the electric field, resulting in a higher effective viscosity. Of considerable interest is the dependence of this effect on the *direction of the electric field*. Towards this end, we propose a *micropolar theory* including appropriate balance and constitutive equations for these suspensions essentially based on the works of Eringen. An appropriate non-dimensionalization is carried out which combines procedures of Eringen for micropolar fluids, on the one hand, and Eckart and Růžička for electrorheological fluids on the other. We then derive constitutive equations for the Cauchy stress and the couple stress and discuss the restrictions imposed on them by the second law of thermodynamics using scaling arguments.

To illustrate the enhanced possibilities of micropolar electrorheology, a simple constitutive model which is linear in the strain rate is discussed in a study of a viscometric flow. We finally show that the velocity profile (hence the flow rate) may strongly depend on the direction of the electric field.

**Key words:** micropolar theory, electrorheological fluids, entropy inequality, non-dimensionalization, constitutive theory, viscometric flow.

## 1. Introduction

Many *electrorheological fluids* (abbreviated: ERFs) are *suspensions* consisting of a solid phase, the particles, and a fluid phase, the carrier oil. These suspensions change their material properties dramatically if they are exposed to an electric field. The observed *increase* of the measured shear stresses (or the measured viscosity) will be called *electrorheological effect* (abbreviated: ERE). It is generally accepted that the ERE of such fluids is essentially due to the existence of particle structures forming in the presence of an electric field hindering the flow and resulting in a higher, apparent viscosity. It is often assumed that these structures are essentially in the direction of the electric field, at least if the fluid is at rest or slowly flowing. For an overview especially of microscopic models and explanations in electrorheology we refer the reader to Parthasarathy and Klingenberg (1996).

Of considerable interest is the dependence of the ERE on the direction of the electric field. If we consider a pressure driven channel flow with a (constant) electric field either perpendicular or parallel to the flow direction, for example, then these structures would also be essentially perpendicular or parallel to the flow direction, respectively. It is to be expected that the ERE is not the same in both cases. Furthermore, in every industrial application the electric field is more or less inhomogeneous (especially the direction is not constant), and thus a precise description of the dependence of the ERE on the direction of the electric field is necessary to develop effective technical applications.

So far, not many attempts have been made to describe this dependence accurately. For example, the general model for the Cauchy stress tensor first described by Rajagopal and Wineman in (1992) and later also used by Ceccio and Wineman in (1994), Eckart in (2000) and Růžička in (2000) is admittedly capable of

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*some* dependence on the direction of the electric field, but it cannot account for different material response, if the (constant) electric field is either perpendicular or parallel to the flow and has the same field strength. Brunn and Abu-Jdayil studied fluids with transverse isotropy (director theory) as models for electrorheological fluids in Brunn and Abu-Jdayil (1998), but with the intention of describing normal yield stresses. Instead of the electric field, they used the director as an independent variable. Unfortunately, no relationship between the electric field and the director is given. Furthermore, their stress is still symmetric.

As was already indicated, the ERF can clearly be viewed as a fluid with microstructure, if an electric field is present. There are various possibilities to describe such fluids, see especially the monograph of Stokes for a comparison of *couple stress fluids*, *anisotropic fluids*, *microfluids and micropolar fluids*, (Stokes, 1984).

The *micropolar theory* was essentially developed by Eringen in a series of papers, from which we name only the most interesting ones in view of our paper: Eringen (1966), (1980), (1997), Kafadar and Eringen (1971). In all papers, balance equations and constitutive equations for microfluids or micropolar fluids are given and discussed. The entropy inequality is evaluated and special cases are investigated. In Eringen (1966), Eringen studied micropolar fluids in a steady, pressure driven pipe flow. He illustrated the velocity, the micro-rotation, the shear stress difference and the couple stress. Kafadar and Eringen studied micropolar media (not only fluids) in Kafadar and Eringen (1971). Anisotropic micropolar fluids were studied in Eringen (1980). In Eringen (1997), Eringen studied liquid crystals subject to electromagnetic fields and discussed various special cases, especially the passage to director theory.

Electromagnetic interactions on micropolar fluids were also studied in Tanahashi and Okanaga (1989). Tanahashi and Okanaga focussed on the derivation of balance equations and discussed the transition from the relativistic case to a non-relativistic case.

The *theory of anisotropic fluids* was essentially developed by Ericksen and Leslie, see for example Ericksen (1960), (1961), (1991), Leslie (1968), (1979). The main difference in the micropolar theory of Eringen is the introduction of a so-called *director* (and an additional equation for it) which describes at least the orientation (possibly also the length) of a particle or molecule. In Ericksen (1960), (1961), Ericksen focussed on the governing equations of anisotropic fluids and applied his theory to steady shear flows of viscoelastic fluids. In Ericksen (1991) he discussed special types of liquid crystals in greater detail.

In addition to what was said up to now, the present paper may be motivated by an experimental observation. Wunderlich (2000) obtained experimental results in a pressure driven channel flow of an electrorheological fluid with particles which lead to the conclusion that the effective viscosity is lower, if the electric field is not perpendicular to the flow, but has also a significant component parallel to the flow. This was obtained by comparing different electrodes in a channel flow.

Both theories mentioned above are surely appropriate to describe this observed phenomenon. However, we have decided to follow essentially the works of Eringen. Thus, we propose a micropolar theory in the present paper that is based on rational thermodynamics and electrodynamics of moving media. This includes the derivation of appropriate balance equations by means of a non-dimensionalization procedure as well as the statement of constitutive equations, both of general and simplified (linearized) form. Finally, we investigate the linearized constitutive equations in a viscometric flow and illustrate the velocity profile, and the flow rate. It can be explicitly shown that these quantities indeed depend on the direction of the electric field in a way that is qualitatively in agreement with the experimental results observed in Wunderlich (2000).

Consequently, this paper is arranged as follows. In section 2, we record the balance laws, the Clausius-Duhem-inequality and Maxwell's equations for a micropolar continuum. This is followed by a discussion of constitutive relations appropriate for ERFs, the restrictions imposed by the invariance requirements and the entropy inequality and some comments on thermodynamic equilibrium. In section 3, in view of electrorheological applications, a non-dimensionalization with a subsequent approximation is carried out. Such procedures were also used in Eckart (2000), Rajagopal and Růžička (1996) and Růžička (2000), for example, however for non-polar electrorheological fluids. In this paper, we follow essentially (Růžička , 2000). The approximation made in Tanahashi and Okanaga (1989) is different from our approach insofar as Tanahashi and Okanaga used the „usual" non-relativistic approximation without introducing a specific non-dimensionalization. In Eringen (1997), no approximation is carried out at all.

Having derived the balance equations for micropolar fluids in the electrorheological approximation, we switch to the constitutive theory in section 4. Here we especially introduce the general constitutive relations of the Cauchy stress and the couple stress, simplify it and evaluate the residual entropy inequality with scaling arguments to obtain certain restrictions on the material parameters. Linear constitutive equations (linear in the velocity gradient and the gradient of the microrotational velocity, respectively) are given for the Cauchy stress and the couple stress, respectively. Constitutive equations for micropolar fluids without electric fields were introduced in Eringen (1966), (1980), while (quasi-linear) constitutive equations for special liquid crystals subject to electromagnetic fields were given in Eringen (1997).

In section 5, a viscometric flow of a (linear) micropolar electrorheological fluid is discussed. We compare shear stresses for different directions of the electric field and show the influence of the different material parameters. Furthermore, the solutions for the velocity and the microrotation are derived explicitly by assuming Dirichlet boundary conditions. Moreover, we illustrate the velocity, and the flow rate depending on the direction and the absolute value of the electric field. Viscometric flows of micropolar fluids have also been considered for example in Eringen (1966), (1980), Stokes (1984).

Finally, we close by summarizing the results in section 6.

### 1.1. Notation/Preliminaries

Let $\Omega$ denote the reference configuration of an abstract body. In a micropolar continuum each material point has three translational and additionally three rotational degrees of freedom, i.e., it is phenomenologically equivalent to a rigid body. The motion of the fluid is determined by a one-to-one mapping $\hat{\boldsymbol{x}}$ that assigns to each material point $\boldsymbol{X} \in \Omega$ a position $\boldsymbol{x}$ in the three dimensional Euclidean space at an instant of time *t*, i.e.,

$$\boldsymbol{x} = \hat{\boldsymbol{x}}(t, \boldsymbol{X}),$$

and a proper orthonormal tensor $\mathrm{X}$, i.e., $\mathrm{X}\mathrm{X}^T = \boldsymbol{I}, \boldsymbol{I}$ is the *identity matrix*, that assigns to each material point $\boldsymbol{X} \in \Omega$ a rotation $\mathrm{X}$ at an instant of time *t*, i.e.,

$$\mathrm{X} = \hat{\mathrm{X}}(t, \boldsymbol{X}).$$

The *material velocity* $\boldsymbol{v}(t, \boldsymbol{x})$, the *velocity gradient* $\boldsymbol{L}(t, \boldsymbol{x})$ and the *microrotational velocity tensor* $\mathrm{W}(t, \boldsymbol{x})$ are defined through

$$\boldsymbol{v} := \frac{\partial \hat{\boldsymbol{x}}}{\partial t}, \qquad \boldsymbol{L} := \nabla \boldsymbol{v} = \left( \frac{\partial \mathrm{v}_i}{\partial x_j} \right)_{i,\, j=1,2,3} \qquad \text{and} \qquad \mathrm{W} := \dot{\mathrm{X}}\, \mathrm{X}^T = -\mathrm{X}\, \dot{\mathrm{X}}^T$$

where $\nabla$ is the derivative with respect to $\boldsymbol{x}$ and where we fixed the standard Cartesian basis $\boldsymbol{e}_i$, $i = 1,2,3$, in order to give a representation of $\nabla \boldsymbol{v}$ in terms of the partial derivatives $\frac{\partial \mathrm{v}_i}{\partial x_j}$, $i, j = 1,2,3$. The *material time derivative* is denoted by $d/dt$ or by a superposed dot. $\varepsilon$ denotes the *isotropic third order tensor*. For vectors $\boldsymbol{u}$, $\boldsymbol{w}$, second order tensors $\boldsymbol{S}$, $\boldsymbol{T}$ we use the notation

$$|\boldsymbol{u}|^2 = u_i u_i\,, \qquad \boldsymbol{u} \cdot \boldsymbol{w} = u_i w_i\,, \qquad \boldsymbol{u} \times \boldsymbol{w} = \left( \varepsilon_{ijk} u_j w_k \right)_{i=1,2,3}, \qquad \boldsymbol{u} \otimes \boldsymbol{w} = \left( u_i w_j \right)_{i,\, j=1,2,3},$$

$$\boldsymbol{S}:\boldsymbol{T}=S_{ij}T_{ij}\,,\qquad \boldsymbol{ST}=\left(S_{ij}T_{jk}\right)_{i,k=1,2,3},\qquad \left|\boldsymbol{S}\right|^2=S_{ij}S_{ji}\,,\qquad \boldsymbol{Su}=\left(S_{ij}u_j\right)_{i=1,2,3},$$

$$\mathrm{e}:\boldsymbol{S}=\left(\varepsilon_{ijk}S_{jk}\right)_{i=1,2,3},\qquad \mathrm{e}\cdot\boldsymbol{u}=\left(\varepsilon_{ijk}u_k\right)_{i,j=1,2,3},$$

in which the summation convention over repeated indices is employed. We will use this convention throughout the paper. For functions $\boldsymbol{u}:\Omega\to\Re^3$, $\boldsymbol{S}:\Omega\to\Re^{3\times3}$ and vectors $\boldsymbol{w}$ we use the notation

$$\mathrm{div}\,\boldsymbol{u}=\frac{\partial u_i}{\partial x_i},\qquad \mathrm{div}\,\boldsymbol{S}=\left(\frac{\partial S_{ij}}{\partial x_j}\right)_{i=1,2,3},\qquad [\nabla\boldsymbol{u}]\boldsymbol{w}=\left(\frac{\partial u_i}{\partial x_j}w_j\right)_{i=1,2,3},$$

while for scalar functions $e:\Re^3\to\Re$, $f:\Re^{3\times3}\to\Re$ we denote the corresponding Fréchet derivatives by

$$\frac{\partial e}{\partial \boldsymbol{w}}=\left(\frac{\partial e}{\partial w_i}\right)_{i=1,2,3},\qquad \frac{\partial f}{\partial \boldsymbol{A}}=\left(\frac{\partial f}{\partial A_{ij}}\right)_{i,j=1,2,3}.$$

The *symmetric part of the velocity gradient* is denoted by $\boldsymbol{D}$ and the *skewsymmetric part* by $\boldsymbol{W}$. A uniquely determined *microrotational velocity vector* $\mathrm{w}$ is associated to $\mathrm{W}$ by

$$\mathrm{w}=-\frac{1}{2}\mathrm{e}:\mathrm{W}\,,\qquad \mathrm{W}=-\mathrm{e}\cdot\mathrm{w}\,.$$

Furthermore, we shall assume sufficient smoothness of all the field variables in order to make all operations that are carried out meaningful. Throughout this paper we use MKSA-units (cf. Jackson, 1983).

## 2. Balance laws

We start by stating Maxwell's equations. Here we use the so-called „statistical formulation", which is based on a „dipole-current-loop" model (see Eringen and Maugin, 1989; Hutter and van de Ven, 1978; Grot, 1976; Pao, 1978)

$$\mathrm{curl}\,\boldsymbol{E}=-\frac{\partial\boldsymbol{B}}{\partial t},\tag{2.1}$$

$$\mathrm{curl}\,\boldsymbol{H}=-\frac{\partial\boldsymbol{D}^e}{\partial t}+\boldsymbol{J}\,,\tag{2.2}$$

$$\mathrm{div}\,\boldsymbol{D}^e=q^e\,,\tag{2.3}$$

$$\mathrm{div}\,\boldsymbol{B}=0\tag{2.4}$$

where $\boldsymbol{E}$ is the *electric field*, $\boldsymbol{B}$ the *magnetic flux density*, $\boldsymbol{H}$ is the *magnetic field* given by $\boldsymbol{H}=\frac{1}{\mu_0}\boldsymbol{B}-\boldsymbol{M}$ with the *magnetization* $\boldsymbol{M}$, $\boldsymbol{D}^e$ is the *dielectric displacement* given by $\boldsymbol{D}^e=\boldsymbol{P}+\varepsilon_0\boldsymbol{E}$ with the *electric* *polarization* $\boldsymbol{P}$, $\boldsymbol{J}$ the *current density*, $q^e$ the *density of the free electric charges* and $\varepsilon_0$ and $\mu_0$ denote the dielectric constant and the permeability in vacuum, respectively.

The balance equations for *micropolar continua* are well known. Here, we follow essentially Eringen (see 1966; 1980; 1997), Kafadar and Eringen (1971). The balance of mass and momentum are

$$\dot{\rho} + \rho \operatorname{div} \boldsymbol{v} = 0\,, \tag{2.5}$$

$$\rho \dot{\boldsymbol{v}} - \operatorname{div} \boldsymbol{T} = \boldsymbol{f} + \boldsymbol{f}^{e}\,, \tag{2.6}$$

respectively, where $\rho$ is the *mass density*, $\boldsymbol{T}$ the Cauchy *stress tensor*[1], $\boldsymbol{f}$ the *mechanical force density* and $\boldsymbol{f}^e$ is the *electromagnetic force density* which is given by (cf. (Pao, 1978, pp.284-285), (Hutter and van de Ven, 1978, p.64-65))

$$\boldsymbol{f}^{e} = q^{e}\mathcal{E} + \left[\mathcal{J} + \dot{\boldsymbol{P}} - [\nabla \boldsymbol{v}]\boldsymbol{P} + (\operatorname{div}\boldsymbol{v})\boldsymbol{P}\right] \times \boldsymbol{B} + [\nabla \boldsymbol{B}]^{T}\mathcal{M} + [\nabla \mathcal{E}]\boldsymbol{P}$$

where $\mathcal{E}$ is the *effective electric field strength* defined as

$$\mathcal{E} = \boldsymbol{E} + \boldsymbol{v} \times \boldsymbol{B}\,, \tag{2.7}$$

$\mathcal{J}$ the *conductive current density* given by

$$\mathcal{J} = \boldsymbol{J} - q^{e}\boldsymbol{v}\,, \tag{2.8}$$

and $\mathcal{M}$ the *effective magnetization* defined through

$$\mathcal{M} = \boldsymbol{M} + \boldsymbol{v} \times \boldsymbol{P}\,. \tag{2.9}$$

The balance of internal spin takes the form

$$\rho \dot{\boldsymbol{s}} - \operatorname{div} \boldsymbol{N} = \varepsilon : \boldsymbol{T}^{T} + \boldsymbol{l} + \boldsymbol{l}^{e} \tag{2.10}$$

where $\boldsymbol{s}$ is the *specific internal spin*, $\boldsymbol{N}$ is the *couple stress tensor*[2], $\boldsymbol{l}$ the *mechanical couple density* and $\boldsymbol{l}^e$ the *electromagnetic couple density* which is given as [cf. (Pao, 1978, pp.284-285), (Hutter and van de Ven, 1978, p.64-65)]

$$\boldsymbol{l}^{e} = \boldsymbol{P} \times \mathcal{E} + \mathcal{M} \times \boldsymbol{B}\,.$$

As usual, we define the *specific internal spin* $\boldsymbol{s}$ according to

$$\boldsymbol{s} = \Theta\omega\,, \tag{2.11}$$

in which the symmetric *micro-inertia tensor* $\Theta$ which represents the microstructure of the fluid fulfills the following relation

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[1] $\boldsymbol{T}$ is introduced via $\boldsymbol{t} = \boldsymbol{T} \cdot \boldsymbol{n}$, where $\boldsymbol{t}$ is the Cauchy *stress vector* and $\boldsymbol{n}$ the *outer unit normal vector*.

[2] $\boldsymbol{N}$ is introduced via $\boldsymbol{m} = \boldsymbol{N} \cdot \boldsymbol{n}$ where $\boldsymbol{m}$ is the *couple stress vector* and $\boldsymbol{n}$ the *outer unit normal vector*.

$$\dot{\mathrm{Q}} - \mathrm{W}\mathrm{Q}^T - \mathrm{Q}\mathrm{W}^T = \boldsymbol{0} . \tag{2.12}$$

With the ***kinetic energy density*** $\frac{1}{2}\rho\boldsymbol{v}\cdot\boldsymbol{v} + \frac{1}{2}\rho\boldsymbol{s}\cdot\mathrm{w}$, the balance of total energy takes the form

$$\rho\frac{d}{dt}\left(e + \frac{1}{2}\boldsymbol{v}\cdot\boldsymbol{v} + \frac{1}{2}\mathrm{w}\cdot\boldsymbol{s}\right) = \mathrm{div}\left(\boldsymbol{T}^T\boldsymbol{v} + \boldsymbol{N}^T\mathrm{w} - \boldsymbol{q}\right) + \left(\boldsymbol{f} + \boldsymbol{f}^e\right)\cdot\boldsymbol{v} + \left(\boldsymbol{1} + \boldsymbol{1}^e\right)\cdot\mathrm{w} + w + w^e \tag{2.13}$$

where $e$ denotes the *specific internal energy*, $\boldsymbol{q}$ the *heat flux*, $w$ the *mechanical energy supply density* and $w^e$ the *electromagnetic energy production density* which is given as [cf. (Pao, 1978, pp.284-285), (Hutter and van de Ven, 1978, p.64-65)]

$$w^e = \mathrm{Á}\cdot\mathrm{E} + \mathrm{E}\cdot\dot{\boldsymbol{P}} - \mathrm{M}\cdot\dot{\boldsymbol{B}} + \mathrm{E}\cdot\boldsymbol{P}\,\mathrm{div}\boldsymbol{v} .$$

From Eq.(2.12), the symmetry of $\mathrm{Q}$ and the skewsymmetry of $\mathrm{Q}$ follows

$$\mathrm{w}\cdot\left(\dot{\mathrm{Q}}\mathrm{w}\right) = \boldsymbol{0} .$$

Thus we obtain from Eq.(2.11)

$$\mathrm{w}\cdot\dot{\boldsymbol{s}} = \mathrm{w}\cdot\left(\dot{\mathrm{Q}}\mathrm{w}\right) + \boldsymbol{s}\cdot\dot{\mathrm{w}} = \boldsymbol{s}\cdot\dot{\mathrm{w}} . \tag{2.14}$$

Multiplying Eq.(2.6) by $\boldsymbol{v}$, Eq.(10) by $\mathrm{w}$ and using Eqs (2.14), (1.1) and (2.2), we obtain from Eq.(2.13) the reduced balance of internal energy according to

$$\rho\dot{e} + \mathrm{div}\boldsymbol{q} = \boldsymbol{T} : \left(\boldsymbol{D} + \boldsymbol{R}\right) + \boldsymbol{N} : \left(\nabla\mathrm{w}\right) + \mathrm{Á}\cdot\mathrm{E} + \mathrm{E}\cdot\dot{\boldsymbol{P}} - \mathrm{M}\cdot\dot{\boldsymbol{B}} + \boldsymbol{P}\cdot\mathrm{E}\,\mathrm{div}\boldsymbol{v} + w \tag{2.15}$$

where we introduced the notation $\boldsymbol{R} = \boldsymbol{W} - \mathrm{W}$.

We interpret the second law of thermodynamics in the form of the Clausius-Duhem inequality[3]:

$$\rho\dot{\eta} + \mathrm{div}\left(\frac{\boldsymbol{q}}{\theta}\right) - \frac{w}{\theta} \geq \boldsymbol{0} \tag{2.16}$$

where $\eta$ is the *specific entropy* and $\theta$ is the *absolute temperature*.

The system (2.1)-(2.4), (2.5), (2.6), (2.10), (2.15) and (2.16) which describes the motion of the body has far more unknowns than equations. It is rendered determinate by providing appropriate constitutive relations reflecting the material properties. Towards this end, we will assume that

$$\rho,\ \theta,\ \mathrm{Q},\ \nabla\theta,\ \boldsymbol{v},\ \boldsymbol{D},\ \boldsymbol{R},\ \left(\nabla\mathrm{w}\right)_S,\ \left(\nabla\mathrm{w}\right)_A,\ \boldsymbol{E},\ \boldsymbol{B} \tag{2.17}$$

where $\left(\nabla\mathrm{w}\right)_S, \left(\nabla\mathrm{w}\right)_A$ - the symmetric and the skewsymmetric parts of $\nabla\mathrm{w}$ are the independent variables and thus we provide constitutive relations for

[3] The second law of thermodynamics can also be evaluated by the method introduced by Liu and Müller (1972), Liu (1972; 2002), for example. We also refer the reader to Hutter (1975; 1977), Hutter and van de Ven (1978), Leslie (1979) and Müller and Ruggeri (1993) for different formulations of the second law of thermodynamics.

$$e, \eta, \boldsymbol{T}, \boldsymbol{N}, \boldsymbol{q}, \boldsymbol{P}, \mathrm{M}, \text{Á}, \tag{2.18}$$

of the form

$$f = \hat{f}\left(\rho, \theta, \mathrm{Q}, \nabla\theta, \boldsymbol{v}, \boldsymbol{D}, \boldsymbol{R}, (\nabla \mathrm{w})_S, (\nabla \mathrm{w})_A, \boldsymbol{E}, \boldsymbol{B}\right) \tag{2.19}$$

where *f* stands for any of the quantities in Eq.(2.18).

Note that one usually introduces the sum $\boldsymbol{D}+\boldsymbol{R}$ and $\nabla \mathrm{w}$ as independent variables [cf. (Eringen, 1999; Leslie, 1979)], while here we use $\boldsymbol{D}$, $\boldsymbol{R}$, $(\nabla \mathrm{w})_S$ (symmetric part of $\nabla \mathrm{w}$) and $(\nabla \mathrm{w})_A$ (skewsymmetric part of $\nabla \mathrm{w}$) separately. This approach allows us to consider constitutive equations that are not of the same order in $\boldsymbol{D}$ and $\boldsymbol{R}$ and $(\nabla \mathrm{w})_S$ and $(\nabla \mathrm{w})_A$, respectively. We will come back to this in section 4.

As pointed out by Eringen in (1980), the micro-inertia tensor $\mathrm{Q}$ enlarges the possibilities of describing new effects remarkably, if it is taken as an independent variable. Indeed, especially for electrorheological fluids that may be considered to have variable micro-inertia, it is recommended to take it into account.

Both the material and the balance equations are subject to invariance requirements. It is well known that the mechanical balance laws Eqs (2.5), (2.6), (2.10) and (2.15) are forminvariant under Galilean transformations given by

$$\boldsymbol{x}^* = \boldsymbol{Q}\boldsymbol{x} + \boldsymbol{v}_0 t + \boldsymbol{b}_0, \qquad t^* = t \tag{2.20}$$

where $\boldsymbol{v}_0$, $\boldsymbol{b}_0$ are constant vectors and $\boldsymbol{Q}$ is a time independent orthogonal tensor, while Maxwell's Eqs (2.1)-(2.4) are forminvariant under Lorentz transformations[4]. We are interested in non-relativistic effects and it is well-known that there are problems with consistent invariance requirements for all thermo-mechanical and electro-magnetic balance laws and constitutive equations in a non-relativistic situation [cf. (Grot, 1976; Rajagopal and Růžička, 1996; Růžička, 2000)]. To avoid these difficulties we shall make the following invariance requirements. We assume that the quantities (2.18), describing the material properties, are invariant under Galilean transformations (2.20)[5]. Moreover we require that all balance laws Eqs (2.5), (2.6), (2.10), (2.15), (2.16) and (2.1)-(2.4) are forminvariant under Galilean transformations (2.20). These two requirements imply consistent transformation formulae for all necessary quantities [cf. (Růžička, 2000)]. In particular, we obtain from the invariance requirements that the constitutive relations Eq.(2.19) are isotropic functions of their arguments and that Eq.(2.19) has to be replaced by [cf. (Grot, 1976)]

$$f = \hat{f}\left(\rho, \theta, \mathrm{Q}, \nabla\theta, \boldsymbol{D}, \boldsymbol{R}, (\nabla \mathrm{w})_S, (\nabla \mathrm{w})_A, \mathrm{E}, \boldsymbol{B}\right) \tag{2.21}$$

where *f* stands for any of the quantities in Eq.(2.18).

We require that the second law of thermodynamics in the form of the Clausius-Duhem inequality (2.16) is satisfid in all admissible processes, i.e., processes compatible with the balance laws Eqs (2.1)-(2.4), (2.5), (2.6), (2.10), (2.15) and constitutive relations for Eq.(2.18) in the form (2.21). This requirement places

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[4] For the rotation $\mathrm{X}$ we require that it transforms according to $\mathrm{X}^* = \boldsymbol{Q}\mathrm{X}$ (cf. (Kafadar and Eringen, 1971)). For the remaining quantities we refer the reader to Růžička (2000), section 1.1.

[5] Note that one usually assumes that the constitutive relations depend on $\boldsymbol{L}$ instead of $\boldsymbol{D}$, and then one deduces from the principle of material frame indifference, i.e., (2.20) is replaced by $\boldsymbol{x}^* = \boldsymbol{Q}(t)\boldsymbol{x} + \boldsymbol{c}(t)$, that the dependence on $\boldsymbol{L}$ has to reduce to a dependence on $\boldsymbol{D}$ only. Moreover, we introduced the quantity $\boldsymbol{R} = \boldsymbol{W} - \mathrm{W}$ because it is invariant under the above transformation but $\boldsymbol{W}$ and $\mathrm{W}$ alone are not. In fact, these are the only relevant consequences of the stronger requirement of material frame indifference for us which cannot be obtained from the requirement that the material properties are invariant under Galilean transformations (2.20) alone.

further restrictions on the form of the constitutive relations. Firstly, introducing the specific free energy $\psi$ through

$$\psi = e - \eta\theta - \frac{1}{\rho}\mathrm{E}\cdot\boldsymbol{P}, \tag{2.22}$$

we re-write Eq.(2.16), using Eqs (2.5) and (2.15), as

$$-\rho\left(\dot{\psi} + \eta\dot{\theta}\right) + \boldsymbol{T}:\left(\boldsymbol{D}+\boldsymbol{R}\right) + \boldsymbol{N}:[\nabla\mathrm{w}] + \acute{\mathrm{A}}\cdot\mathrm{E} - \mathsf{M}\cdot\dot{\boldsymbol{B}} - \dot{\mathrm{E}}\cdot\boldsymbol{P} - \frac{\boldsymbol{q}\cdot\nabla\theta}{\theta} \geq 0\,. \tag{2.23}$$

Using Eq.(2.21) and computing $\dot{\psi}$ in Eq.(2.23) we obtain using again Eq.(2.5)

$$\begin{aligned}&\left(\boldsymbol{T} + \rho^2\frac{\partial\psi}{\partial\rho}\boldsymbol{I}\right):\boldsymbol{D} - \rho\left(\frac{\partial\psi}{\partial\theta} + \eta\right)\dot{\theta} - \rho\frac{\partial\psi}{\partial\mathrm{Q}}:\left(\mathrm{W}\mathrm{Q}^T + \mathrm{Q}\mathrm{W}^T\right) - \rho\frac{\partial\psi}{\partial\nabla\theta}\left(\nabla\theta\right)^{\cdot} + \\ &-\rho\frac{\partial\psi}{\partial\boldsymbol{D}}\cdot\dot{\boldsymbol{D}} - \rho\frac{\partial\psi}{\partial\boldsymbol{R}}\cdot\dot{\boldsymbol{R}} - \rho\frac{\partial\psi}{\partial(\nabla\mathrm{w})_S}\cdot(\nabla\mathrm{w})_S^{\cdot} - \rho\frac{\partial\psi}{\partial(\nabla\mathrm{w})_A}\cdot(\nabla\mathrm{w})_A^{\cdot} + \\ &-\left(\rho\frac{\partial\psi}{\partial\mathrm{e}} + \boldsymbol{P}\right)\cdot\dot{\mathrm{E}} - \left(\mathsf{M} + \rho\frac{\partial\psi}{\partial\boldsymbol{B}}\right)\cdot\dot{\boldsymbol{B}} + \boldsymbol{T}:\boldsymbol{R} + \boldsymbol{N}:[\nabla\mathrm{w}] + \acute{\mathrm{A}}\cdot\mathrm{E} - \frac{\boldsymbol{q}\cdot\nabla\theta}{\theta} \geq 0.\end{aligned} \tag{2.24}$$

Since at every fixed point in space and every instant in time the quantities

$$\dot{\theta},\ (\nabla\theta)^{\cdot},\ \dot{\boldsymbol{D}},\ \dot{\boldsymbol{R}},\ (\nabla\mathrm{w})_S^{\cdot},\ (\nabla\mathrm{w})_A^{\cdot},\ \dot{\mathrm{E}},\ \dot{\boldsymbol{B}},$$

can be chosen arbitrarily and are independent of the arguments in Eq.(2.21) (cf. Coleman and Noll (1963), Truesdell and Noll (1965)) and since the inequality (2.24) is linear in Eq.(2.2) we immediately deduce the following relations

$$\begin{aligned}&\frac{\partial\psi}{\partial\nabla\theta} = \boldsymbol{0}, \quad \frac{\partial\psi}{\partial\boldsymbol{D}} = \boldsymbol{0}, \quad \frac{\partial\psi}{\partial\boldsymbol{R}} = \boldsymbol{0}, \quad \frac{\partial\psi}{\partial(\nabla\mathrm{w})_S} = \boldsymbol{0}, \quad \frac{\partial\psi}{\partial(\nabla\mathrm{w})_A} = \boldsymbol{0}, \\ &\eta = -\frac{\partial\psi}{\partial\theta}, \qquad \boldsymbol{P} = -\rho\frac{\partial\psi}{\partial\mathrm{E}}, \qquad \mathsf{M} = -\rho\frac{\partial\psi}{\partial\boldsymbol{B}}.\end{aligned} \tag{2.25}$$

Thus, $\psi, \eta$, $\boldsymbol{P}$ and $\mathsf{M}$ are functions of $\rho, \theta, \mathrm{Q}, \mathrm{E}$ and $\boldsymbol{B}$ only. Furthermore, the following residual entropy inequality remains

$$\left(\boldsymbol{T} + \rho^2\frac{\partial\psi}{\partial\rho}\boldsymbol{I}\right):\boldsymbol{D} + \boldsymbol{T}:\boldsymbol{R} + \boldsymbol{N}:\nabla\mathrm{w} + \acute{\mathrm{A}}\cdot\mathrm{E} - \rho\frac{\partial\psi}{\partial\mathrm{Q}}:\left(\mathrm{W}\mathrm{Q}^T + \mathrm{Q}\mathrm{W}^T\right) - \frac{\boldsymbol{q}\cdot\nabla\theta}{\theta} \geq 0\,. \tag{2.26}$$

We can derive further restrictions of the constitutive relations by evaluating the residual entropy inequality (2.26). Following (Liu and Müller, 1972; Nečas and Šilhavý, 1991; Růžička, 1992) we introduce the equilibrium part of $\boldsymbol{T}$, $\boldsymbol{N}$, $\boldsymbol{q}$, $\acute{\mathrm{A}}$ and $\psi$ through (cf. Hutter (1977))

$$f^E(\rho, \theta, \mathrm{Q}, \boldsymbol{B}) := \hat{f}(\rho, \theta, \mathrm{Q}, \boldsymbol{0}, \boldsymbol{0}, \boldsymbol{0}, \boldsymbol{0}, \boldsymbol{0}, \boldsymbol{0}, \boldsymbol{B}),$$

$$\psi^E(\rho, \theta, \mathrm{Q}, \boldsymbol{B}) := \hat{\psi}(\rho, \theta, \mathrm{Q}, \boldsymbol{0}, \boldsymbol{B}),$$

and the non-equilibrium parts through

$$f^D(\rho, \theta, \mathrm{Q}, \nabla\theta, \boldsymbol{D}, \boldsymbol{R}, (\nabla \mathrm{w})_S, (\nabla \mathrm{w})_A, \mathrm{E}, \boldsymbol{B}) := f(\rho, \theta, \mathrm{Q}, \nabla\theta, \boldsymbol{D}, \boldsymbol{R}, (\nabla \mathrm{w})_S, (\nabla \mathrm{w})_A, \mathrm{E}, \boldsymbol{B}) + \\ - f^E(\rho, \theta, \mathrm{Q}, \boldsymbol{B}),$$

$$\psi^D(\rho, \theta, \mathrm{Q}, \mathrm{E}, \boldsymbol{B}) := \psi(\rho, \theta, \mathrm{Q}, \mathrm{E}, \boldsymbol{B}) - \psi^E(\rho, \theta, \mathrm{Q}, \boldsymbol{B})$$

where *f* stands for $\boldsymbol{T}$, $\boldsymbol{N}$, $\boldsymbol{q}$ and Á. Since Eq.(2.26) has to hold in every process, we consider, at a fixed point $x_0$ in space and a fixed instant $t_0$ in time, the process $\rho, \theta, \boldsymbol{v}, \mathrm{w}, \mathrm{Q}, \boldsymbol{B}, \boldsymbol{E}$ and for $\alpha \in [0, 1]$ the related process $\overline{\rho}, \overline{\theta}, \overline{\boldsymbol{v}}, \overline{\mathrm{w}}, \overline{\mathrm{Q}}, \overline{\boldsymbol{B}}, \overline{\boldsymbol{E}}$ defined through

$$\overline{\rho}(t, x) = \rho(t, x), \qquad \overline{\theta}(t, x) = \theta(t, (1-\alpha)x_0 + \alpha x), \qquad \overline{\boldsymbol{v}}(t, x) = \alpha \boldsymbol{v}(t, x),$$

$$\overline{\mathrm{w}} = \alpha \mathrm{w}(t, x), \qquad \overline{\mathrm{Q}}(t, x) = \mathrm{Q}(t, x),$$

$$\overline{\boldsymbol{B}}(t, x) = \boldsymbol{B}((1-\alpha)t_0 + \alpha t, x), \qquad \overline{\boldsymbol{E}}(t, x) = \alpha \boldsymbol{E}(t, x).$$

Thus, the variables

$$\rho, \theta, \mathrm{Q}, \nabla\theta, \boldsymbol{D}, \boldsymbol{R}, (\nabla \mathrm{w})_S, (\nabla \mathrm{w})_A, \mathrm{E}, \boldsymbol{B}, \tag{2.27}$$

can be systematically replaced by

$$\rho, \theta, \mathrm{Q}, \alpha\nabla\theta, \alpha\boldsymbol{D}, \alpha\boldsymbol{R}, \alpha(\nabla \mathrm{w})_S, \alpha(\nabla \mathrm{w})_A, \alpha\mathrm{E}, \boldsymbol{B}. \tag{2.28}$$

The residual entropy inequality Eq.(2.26) in the new process reads

$$\left(\boldsymbol{T} + \rho^2 \frac{\partial\psi}{\partial\rho}\boldsymbol{I}\right) : \alpha\boldsymbol{D} + \boldsymbol{T} : \alpha\boldsymbol{R} + \boldsymbol{N} : \alpha\nabla \mathrm{w} + \text{Á} \cdot \alpha\mathrm{E} + \\ -\rho\frac{\partial\psi}{\partial\mathrm{Q}} : \left(\alpha \mathrm{W}\,\mathrm{Q}^T + \mathrm{Q}\,\alpha\mathrm{W}^T\right) - \frac{\alpha(\nabla\theta)\cdot\boldsymbol{q}}{\theta} \geq 0$$

where $\boldsymbol{T}$, $\boldsymbol{N}$, $\boldsymbol{q}$ and Á are evaluated by Eq.(2.28) and $\psi$ is evaluated at $\rho, \theta, \mathrm{Q}, \alpha\mathrm{E}, \boldsymbol{B}$. Dividing this inequality by $\alpha$ and letting $\alpha \to 0$ we arrive at

$$\left(\boldsymbol{T}^E + \rho^2 \frac{\partial\psi^E}{\partial\rho}\boldsymbol{I}\right) : \boldsymbol{D} + \boldsymbol{T}^E : \boldsymbol{R} + \boldsymbol{N}^E : \nabla \mathrm{w} + \text{Á}^E \cdot \mathrm{E} + \\ -\rho\frac{\partial\psi^E}{\partial\mathrm{Q}} : \left(\mathrm{W}\,\mathrm{Q}^T + \mathrm{Q}\,\mathrm{W}^T\right) - \frac{(\nabla\theta)\cdot\boldsymbol{q}^E}{\theta} \geq 0.$$

Since the constitutive quantities are evaluated at $\rho, \theta, \mathrm{Q}, \boldsymbol{B}$ this inequality is linear in $\nabla\theta, \boldsymbol{D}, \boldsymbol{R}, \nabla \mathrm{w}, \mathrm{E}$ and $\mathrm{W}$, which can be chosen arbitrarily. Therefore we obtain

$$\boldsymbol{T}^E = -\rho^2 \frac{\partial \psi^E}{\partial \rho} \boldsymbol{I}, \qquad \boldsymbol{T}^E = \left(\boldsymbol{T}^E\right)^T, \qquad \boldsymbol{N}^E = \boldsymbol{0},$$

$$\text{Á}^E = \boldsymbol{0}, \qquad \boldsymbol{q}^E = \boldsymbol{0}, \qquad \frac{\partial \psi^E}{\partial \mathrm{Q}} = \boldsymbol{0}, \tag{2.29}$$

and the residual entropy inequality

$$\left(\boldsymbol{T}^D + \rho^2 \frac{\partial \psi^D}{\partial \rho} \boldsymbol{I}\right) : \boldsymbol{D} + \boldsymbol{T}^D : \boldsymbol{R} + \boldsymbol{N}^D : \nabla \mathrm{w} + \text{Á}^D \cdot \mathrm{E} +$$
$$-\rho \frac{\partial \psi^D}{\partial \mathrm{Q}} : \left(\mathrm{W}\,\mathrm{Q}^T + \mathrm{Q}\,\mathrm{W}^T\right) - \frac{(\nabla\theta)\cdot \boldsymbol{q}^D}{\theta} \geq 0. \tag{2.30}$$

Obviously, the relations Eqs (2.29) and (2.30) are equivalent to Eq.(2.26).

Because the free energy $\psi$ may depend on $\mathrm{Q}$, also the electric polarization and the effective magnetic field strength may depend on this quantity. This emphasizes the importance of the micro-inertia, providing the possibility that effects of microstructure entering Maxwell's equations through $\boldsymbol{P}$ and $\mathsf{M}$.

## 3. Electrorheological approximation

The equations derived in the last section may be simplifed in view of electrorheological applications. To this end, it is recommended to carry out an appropriate nondimensionalization with a subsequent approximation. Note that the electorheological approximation given here differs from well-known non-relativistic approximations insofar, as in the former case we need additional assumptions concerning the magnetic quantities. These and all other assumptions made in this section are based upon our understanding of the behaviour of electrorheological fluids, both from a theoretical and an experimental point of view (cf. (Bloodworth, 1994; Bloodworth and Wendt, 1996; Eckart, 2000; Růžička, 2000; Wunderlich, 2000)).

*Firstly*, we shall assume that the Cauchy stress tensor $\boldsymbol{T}$ and the couple stress tensor $\boldsymbol{N}$ do not depend on the magnetic flux density $\boldsymbol{B}$, i.e.,

$$\boldsymbol{T} = \hat{\boldsymbol{T}}\left(\rho, \theta, \mathrm{Q}, \nabla\theta, \boldsymbol{D}, \boldsymbol{R}, (\nabla \mathrm{w})_S, (\nabla \mathrm{w})_A, \mathrm{E}\right), \tag{3.1}$$

$$\boldsymbol{N} = \hat{\boldsymbol{N}}\left(\rho, \theta, \mathrm{Q}, \nabla\theta, \boldsymbol{D}, \boldsymbol{R}, (\nabla \mathrm{w})_S, (\nabla \mathrm{w})_A, \mathrm{E}\right). \tag{3.2}$$

This assumption reflects the observation that the material properties of an ERF do not change if a magnetic field is applied, because, surely, the particles in an ERF bear no magnetic properties.

*Secondly*, we shall assume that

$$\mathsf{M} \equiv \boldsymbol{0} \qquad \text{where} \qquad \mathsf{M} = \boldsymbol{M} + \boldsymbol{v} \times \boldsymbol{P}, \tag{3.3}$$

$\boldsymbol{M}$ is the *magnetization* and $\boldsymbol{P}$ the electric polarization. This assumption ensures that an apparent magnetization can only be generated by a moving polarized fluid, see also (Grot, 1976). This common assumption is one crucial point to derive the so-called „quasi-electrostatic equations". In view of Kirwan

(1986) assumption Müller (1985) also implies that the specific free energy $\tilde{\psi}$, and thus also the polarization $\boldsymbol{P}$ and the entropy $\eta$, are only functions of $\rho, \theta, \mathrm{Q}$ and $\mathrm{E}$.

*Thirdly,* we shall assume that the fluid is electrically non-conducting, i.e.,

$$Á \equiv \boldsymbol{0}\,. \tag{3.4}$$

This assumption may not be fully justified in general, because some electrorheological fluids exhibit a certain electrical conductivity which is often due to the content of water. However, many of them are free of water and have very low electrical conductivity (for example the polyurethane dispersions described in detail in Bloodworth (1994) and Bloodworth and Wendt (1996)), and thus we may restrict ourselves to such a class. Note that in electrorheological applications like valves and dampers, the effective electric current should be as small as possible to guarantee a small power consumption and to avoid Joule heating.

In order to reach the final electrorheological approximation and to determine and retain terms that are dominant and discard others that are insignificant, we will carry out a dimensional analysis which follows closely that performed in Rajagopal and Růžička (1996), Růžička (2000).

To this end we may introduce the following dimensionless quantities[6]

$$\overline{\boldsymbol{E}} = \frac{\boldsymbol{E}}{E_0}, \quad \overline{\boldsymbol{B}} = \frac{\boldsymbol{B}}{B_0}, \quad \overline{q}_e = \frac{q_e}{q_0}, \quad \overline{\boldsymbol{T}} = \frac{\boldsymbol{T}}{T_0}, \quad \overline{\boldsymbol{N}} = \frac{\boldsymbol{N}}{N_0}, \quad \overline{\mathrm{Q}} = \frac{\mathrm{Q}}{\Theta_0},$$

$$\overline{\boldsymbol{v}} = \frac{\boldsymbol{v}}{V_0}, \quad \overline{\boldsymbol{x}} = \frac{\boldsymbol{x}}{L_0}, \quad \overline{t} = \frac{t}{t_0}, \quad \overline{\boldsymbol{P}} = \frac{\boldsymbol{P}}{\varepsilon_0 E_0}, \quad \overline{\rho} = \frac{\rho}{\rho_0}, \quad \overline{\boldsymbol{f}} = \frac{\boldsymbol{f}}{f_0}, \quad \overline{\theta} = \frac{\theta}{\theta_0}$$

where the quantities with the subscript „*0*" are appropriate characteristic quantities of the problem in question. In typical problems and for many electrorheological fluids (cf. (Bloodworth, 1994 and Bloodworth and Wendt, 1996)), we envisage that

$$\begin{aligned} &E_0 \sim 3\cdot\left(10^4 - 10^6\right)V\,m^{-1}, && V_0 \sim \left(10^{-3} - 1\right)ms^{-1}, \\ &L_0 \sim 5\cdot\left(10^{-4} - 10^{-3}\right)m, && \eta_0 \sim \left(10^{-2} - 10^{-1}\right)kg(m\,s)^{-1}, \\ &t_0 \sim \left(10^{-3} - 1\right)s, && \rho_0 \sim 10^3\,kg\,m^{-3}. \end{aligned} \tag{3.5}$$

The time $t_0$ may either be a characteristic electric or a hydrodynamic time, depending on the specific problem. Moreover, $\rho_0$ and $\eta_0$ are the density and the dynamic viscosity of the fluid in the absence of an electric field, respectively. Using Eq.(3.5), the Reynolds number $\mathrm{Re} = (\rho_0 L_0 V_0)/\eta_0$ and the Strouhal number $\mathrm{Str} = L_0/(V_0 t_0)$ lie in the range

$$5\cdot 10^{-3} \le \mathrm{Re} \le 5\cdot 10^2 \qquad \text{and} \qquad 5\cdot 10^{-4} \le \mathrm{Str} \le 5\cdot 10^3,$$

respectively. Magnetic quantities are missing in Eq.(3.5). No experimental observation is known to us that shows that the magnetic field plays a significant role in electrorheological applications. Usually, no external magnetic field is applied and thus $\boldsymbol{B}$ is only induced by the electric field. We interpret the secondary role of $\boldsymbol{B}$ in electrorheological fluids through the assumption that

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[6] In this section, dimensionless quantities and operators are denoted by a superposed bar.

$$\frac{E_0}{B_0}\frac{L_0}{c^2 t_0} = O(1), \tag{3.6}$$

resulting in

$$B_0 \sim \left(10^{-16} - 10^{-10}\right) V\,s\,m^{-2}.$$

Recall that $c \approx 3 \cdot 10^8\, m\,s^{-1}$ denotes the speed of electromagnetic waves in vacuum. Eqation (3.6) is consistent with the assumption that the magnetic flux density is only induced by oscillations of the electric field and/or the motion of a polarized body (cf. Tanahashi and Okanaga (1989)).

Let us introduce a small non-dimensional number $\varepsilon$ through

$$\varepsilon \equiv 10^{-3},$$

which measures the importance of the terms. The situation described above – together with an assumption that there are only few free charges in the fluid – can thus be summarized as

$$\frac{L_0}{c\,t_0} = O\left(\varepsilon^3\right) - O\left(\varepsilon^4\right), \qquad \frac{V_0}{c} = O\left(\varepsilon^3\right) - O\left(\varepsilon^4\right),$$

$$\frac{V_0 t_0}{L_0} = O\left(\varepsilon^{-1}\right) - O(\varepsilon), \qquad \frac{q_0 L_0}{\varepsilon_0 E_0} = O\left(\varepsilon^3\right), \tag{3.7}$$

$$\frac{B_0 L_0}{E_0 t_0} = O\left(\varepsilon^5\right) - O\left(\varepsilon^8\right), \qquad \frac{E_0 V_0}{B_0 c^2} = O(1).$$

The non-dimensionalized system of balance laws may then be approximated by retaining terms up to order $\varepsilon^2$, while neglecting terms of higher order.

Firstly, let us discuss the role of $\mathrm{E}$ in the constitutive relations. It follows from the definition of $\mathrm{E}$, that

$$\overline{\mathrm{E}} = \frac{\mathrm{E}}{E_0} = \overline{\boldsymbol{E}} + \frac{V_0 B_0}{E_0}\bar{\boldsymbol{v}} \times \overline{\boldsymbol{B}} = \overline{\boldsymbol{E}} + O\left(\varepsilon^5\right) \tag{3.8}$$

where we used that

$$\frac{V_0 B_0}{E_0} = O\left(\varepsilon^5\right) - O\left(\varepsilon^7\right). \tag{3.9}$$

Thus, $\overline{\mathrm{E}}$ can be replaced by $\overline{\boldsymbol{E}}$ in all non-dimensionalized constitutive relations.

The dimensionless form of Maxwell's Eqs (2.1)-(2.4) may be obtained by using the definition of $\boldsymbol{H}$, $\boldsymbol{D}^e$, Eqs (3.3), (3.7) and (3.8)

$$\bar{\operatorname{div}}\,\bar{\boldsymbol{E}} + \bar{\operatorname{div}}\,\bar{\boldsymbol{P}} = \underbrace{\frac{q_0 L_0}{\varepsilon_0 E_0}}_{O(\varepsilon^3)} \bar{q}_e + O(\varepsilon^5), \qquad \bar{\operatorname{curl}}\,\bar{\boldsymbol{E}} + \underbrace{\frac{B_0 L_0}{E_0 t_0}}_{O(\varepsilon^5)} \frac{\partial \bar{\boldsymbol{B}}}{\partial \bar{t}} = \boldsymbol{0}, \qquad \bar{\operatorname{div}}\,\bar{\boldsymbol{B}} = 0,$$

$$\bar{\operatorname{curl}}\,\bar{\boldsymbol{B}} + \underbrace{\frac{E_0 V_0}{B_0 c^2}}_{O(1)} \bar{\operatorname{curl}}(\bar{\boldsymbol{v}} \times \bar{\boldsymbol{P}}) = \underbrace{\frac{E_0}{B_0}\frac{L_0}{c^2 t_0}}_{O(1)} \frac{\partial}{\partial \bar{t}}(\bar{\boldsymbol{E}} + \bar{\boldsymbol{P}}) - \underbrace{\frac{q_0 L_0}{\varepsilon_0 E_0}\frac{E_0 V_0}{B_0 c^2}}_{O(\varepsilon^3)} \bar{q}_e \bar{\boldsymbol{v}} + O(\varepsilon^5)$$

where we also used the relation $\varepsilon_0 \mu_0 = c^{-2}$. Neglecting terms of $O(\varepsilon^3)$, we obtain the electrorheological approximation of Maxwell's equations according to[7]

$$\operatorname{div}(\varepsilon_0 \boldsymbol{E} + \boldsymbol{P}) = 0, \tag{3.10}$$

$$\operatorname{curl} \boldsymbol{E} = 0, \tag{3.11}$$

$$\operatorname{div} \boldsymbol{B} = 0, \tag{3.12}$$

$$\frac{1}{\mu_0} \operatorname{curl} \boldsymbol{B} + \operatorname{curl}(\boldsymbol{v} \times \boldsymbol{P}) = \frac{\partial(\varepsilon_0 \boldsymbol{E} + \boldsymbol{P})}{\partial t} \tag{3.13}$$

where $\quad \boldsymbol{P} = \boldsymbol{P}(\rho, \theta, \mathrm{Q}, \boldsymbol{E})$.

Now we turn to the approximation of the thermo-mechanical balance laws. The conservation of mass Eq.(2.5) remains unaffected. In the momentum Eq.(2.6) we rewrite the electromagnetic force $\boldsymbol{f}^e$ on using (2.7), (3.3), (3.4), (3.7), (3.8) and (3.9), which leads to

$$\begin{aligned} &\frac{\rho_0 V_0 L_0}{\varepsilon_0 E_0^2 t_0} \bar{\rho} \frac{\partial \bar{\boldsymbol{v}}}{\partial \bar{t}} + \frac{\rho_0 V_0^2}{\varepsilon_0 E_0^2} \bar{\rho} [\bar{\nabla} \bar{\boldsymbol{v}}] \bar{\boldsymbol{v}} - \frac{T_0}{\varepsilon_0 E_0^2} \bar{\operatorname{div}}\,\bar{\boldsymbol{T}} = \\ &= f_0 \frac{L_0}{\varepsilon_0 E_0^2} \bar{\boldsymbol{f}} + \underbrace{\frac{q_0 L_0}{\varepsilon_0 E_0}}_{O(\varepsilon^3)} \left( \bar{q}_e \bar{\boldsymbol{E}} + \underbrace{\frac{V_0 B_0}{E_0}}_{O(\varepsilon^5)} \bar{q}_e \bar{\boldsymbol{v}} \times \bar{\boldsymbol{B}} \right) + \underbrace{\frac{B_0 L_0}{E_0 t_0}}_{O(\varepsilon^5)} \frac{\partial \bar{\boldsymbol{P}}}{\partial \bar{t}} \times \bar{\boldsymbol{B}} + \\ &+ \underbrace{\frac{V_0 B_0}{E_0}}_{O(\varepsilon^5)} \left( [\bar{\nabla} \bar{\boldsymbol{P}}] \bar{\boldsymbol{v}} + (\bar{\operatorname{div}}\,\bar{\boldsymbol{v}}) \bar{\boldsymbol{P}} \times \bar{\boldsymbol{B}} + \bar{\boldsymbol{v}} \times ([\bar{\nabla} \bar{\boldsymbol{B}}] \bar{\boldsymbol{P}}) \right) + [\bar{\nabla} \bar{\boldsymbol{E}}] \bar{\boldsymbol{P}} + O(\varepsilon^5) \end{aligned} \tag{3.14}$$

where in $O(\varepsilon^5)$ only terms coming from Eq.(3.8) are included. This form of the nondimensionalization was chosen in order to evaluate the relative importance of the various terms that occur in the electromagnetic force density $\boldsymbol{f}^e$. We see that all underbraced terms on the right-hand side of Eq.(3.14) have to be neglected.

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[7] Since $\mathsf{M} = \boldsymbol{0}$, we can rewrite Eqs (3.10)-(3.13) in terms of $\boldsymbol{E}$, $\boldsymbol{B}$, $\boldsymbol{H}$, $\boldsymbol{D}^e$ only.

We shall retain the mechanical force term and the term with the Cauchy stress. Furthermore, one easily computes that

$$\frac{\rho_0 V_0 L_0}{\varepsilon_0 E_0^2 t_0} = \begin{cases} O(1) - O\left(\varepsilon^2\right) & \text{if} \quad E_0^2 \sim 9\cdot 10^{12} \quad V^2 m^{-2}, \\ O\left(\varepsilon^{-1}\right) - O\left(\varepsilon^1\right) & \text{if} \quad E_0^2 \sim 9\cdot 10^{10} \quad V^2 m^{-2}, \\ O\left(\varepsilon^{-2}\right) - O(1) & \text{if} \quad E_0^2 \sim 9\cdot 10^{8} \quad V^2 m^{-2}, \end{cases} \tag{3.15}$$

$$\frac{\rho_0 V_0^2}{\varepsilon_0 E_0^2} = \begin{cases} O(1) - O\left(\varepsilon^2\right) & \text{if} \quad E_0^2 \sim 9\cdot 10^{12} \quad V^2 m^{-2}, \\ O\left(\varepsilon^{-1}\right) - O\left(\varepsilon^1\right) & \text{if} \quad E_0^2 \sim 9\cdot 10^{10} \quad V^2 m^{-2}, \\ O\left(\varepsilon^{-2}\right) - O(1) & \text{if} \quad E_0^2 \sim 9\cdot 10^{8} \quad V^2 m^{-2}. \end{cases} \tag{3.16}$$

Therefore also the first and second term on the left-hand side of Eq.(3.14) has to be kept. With regard to the approximation of the other thermo-mechanical nondimensionalized equations, we only replace $\overline{\mathrm{E}}$ by $\overline{\boldsymbol{E}}$ since we have no indication of the behaviour of the other quantities.

Therefore, the *electrorheological approximation of the thermo-mechanical balance laws* is given by

$$\dot{\rho} + \rho \operatorname{div} \boldsymbol{v} = 0\,, \tag{3.17}$$

$$\rho \dot{\boldsymbol{v}} - \operatorname{div} \boldsymbol{T} = \boldsymbol{f} + [\nabla \boldsymbol{E}]\boldsymbol{P}\,, \tag{3.18}$$

$$\rho \dot{\boldsymbol{s}} + \operatorname{div} \boldsymbol{N} = \mathrm{e} : \boldsymbol{T}^T + \boldsymbol{l} + \boldsymbol{P} \times \boldsymbol{E}\,, \tag{3.19}$$

$$\dot{\mathrm{Q}} - \mathrm{W}\mathrm{Q}^T - \mathrm{Q}\mathrm{W}^T = \boldsymbol{0}\,, \tag{3.20}$$

$$c_{\mathrm{v}} \rho \dot{\theta} - k \operatorname{div} \nabla\theta - \left( \frac{\partial^2 \tilde{\psi}}{\partial \boldsymbol{E}\, \partial\theta} \cdot \dot{\boldsymbol{E}} - \rho^2 \frac{\partial^2 \tilde{\psi}}{\partial\rho\, \partial\theta} \operatorname{tr} \boldsymbol{D} \right) \theta = \boldsymbol{T} : \boldsymbol{D} + \rho^2 \frac{\partial \tilde{\psi}}{\partial \rho} \operatorname{tr} \boldsymbol{D} + w\,, \tag{3.21}$$

$$\left(\boldsymbol{T}^D - \pi \boldsymbol{I}\right) : \boldsymbol{D} + \boldsymbol{T}^D : \boldsymbol{R} + \boldsymbol{N}^D : \nabla \mathrm{w} - \rho \frac{\partial \psi^D}{\partial \mathrm{Q}} : \left(\mathrm{W}\mathrm{Q}^T + \mathrm{Q}\mathrm{W}^{\mathrm{T}}\right) - \frac{(\nabla\theta)\cdot \boldsymbol{q}^D}{\theta} \geq 0 \tag{3.22}$$

where we used the definition of the *specific heat* $c_{\mathrm{v}}$ and of the *thermodynamic pressure* $\pi$ according to

$$c_{\mathrm{v}} = -\theta \frac{\partial^2 \tilde{\psi}}{\partial \theta^2}\,, \qquad \pi = -\rho^2 \frac{\partial \tilde{\psi}^D}{\partial \rho}\,.$$

Moreover $c_{\mathrm{v}}, \boldsymbol{P}, \pi, \tilde{\psi}$ and $\tilde{\psi}^D$ are functions of $\rho, \theta, \mathrm{Q}$ and $\boldsymbol{E}$; while we have for the dissipative part of the stress tensor $\boldsymbol{T}^D = \boldsymbol{T}^D\left(\rho, \theta, \mathrm{Q}, \nabla\theta, \boldsymbol{D}, \boldsymbol{R}, (\nabla \mathrm{w})_S, (\nabla \mathrm{w})_A, \boldsymbol{E}\right)$ and the dissipative part of the couple stress tensor $\boldsymbol{N}^D = \boldsymbol{N}^D\big(\rho, \theta, \mathrm{Q}, \nabla\theta, \boldsymbol{D}, \boldsymbol{R}, (\nabla \mathrm{w})_S, (\nabla \mathrm{w})_A, \boldsymbol{E}\big)$. In the next section, we will discuss various formulations for the stress $\boldsymbol{T}^D$ and the couple stress tensor $\boldsymbol{N}^D$.

## 4. Constitutive relations

Now we will develop constitutive models for the Cauchy and the couple stress. The models presented should describe the directional dependence of the material response more accurately than previous ones. Nevertheless, in order to keep the already very long and complicated formulae as simple as possible we shall drop the dependence of the dependent variables (2.18) on the micro-inertia tensor $\mathrm{Q}$. Moreover, we keep the dependence on $\nabla\theta$ only in the constitutive relation for $\boldsymbol{q}$ and assume

$$\boldsymbol{q} = -k\nabla\theta$$

where the *thermal conductivity k* is a positive constant. In all other constitutive relations we drop the dependence on $\nabla\theta$. We also restrict ourselves to the case of an incompressible ERF, i.e.,

$$tr\,\boldsymbol{D} = 0.$$

Thus, in view of Eq.(4.17), we can drop in all constitutive relations the dependence on $\rho$, in particular $\boldsymbol{T}^E = \boldsymbol{0}$. For the stress tensor, which may be split according to $\boldsymbol{T}^D = -\pi\boldsymbol{I} + \boldsymbol{S}$, we assume that the *extra stress tensor* ***S*** is of the form

$$\boldsymbol{S} = \boldsymbol{S}(\theta, \boldsymbol{D}, \boldsymbol{R}, \boldsymbol{E}).$$

Note that we drop the dependence of ***S*** on $\nabla \mathrm{w}$ for the sake of simplicity.

From representation theorems (cf. appendix of Eringen and Maugin, (1989) and references therein) it follows that the most general form for ***S*** is given by

$$\begin{aligned}
S &= \alpha_2 \boldsymbol{E}\otimes\boldsymbol{E} + \alpha_3\boldsymbol{D} + \alpha_4\boldsymbol{D}^2 + \alpha_5(\boldsymbol{DE}\otimes\boldsymbol{E} + \boldsymbol{E}\otimes\boldsymbol{DE}) + \alpha_6\big(\boldsymbol{D}^2\boldsymbol{E}\otimes\boldsymbol{E} + \boldsymbol{E}\otimes\boldsymbol{D}^2\boldsymbol{E}\big) + \\
&+ \alpha_7\boldsymbol{R}^2 + \alpha_8(\boldsymbol{DR} - \boldsymbol{RD}) + \alpha_9(\boldsymbol{RE}\otimes\boldsymbol{E} + \boldsymbol{E}\otimes\boldsymbol{RE}) + \alpha_{10}\boldsymbol{RE}\otimes\boldsymbol{RE} + \\
&+ \alpha_{11}\boldsymbol{RDR} + \alpha_{12}\big(\boldsymbol{D}^2\boldsymbol{R} - \boldsymbol{RD}^2\big) + \alpha_{13}\big(\boldsymbol{RDR}^2 - \boldsymbol{R}^2\boldsymbol{DR}\big) + \\
&+ \alpha_{14}\big(\boldsymbol{RE}\otimes\boldsymbol{R}^2\boldsymbol{E} + \boldsymbol{R}^2\boldsymbol{E}\otimes\boldsymbol{RE}\big) + \alpha_{15}\boldsymbol{R} + \alpha_{16}(\boldsymbol{DR} + \boldsymbol{RD}) + \\
&+ \alpha_{17}(\boldsymbol{E}\otimes\boldsymbol{DE} - \boldsymbol{DE}\otimes\boldsymbol{E}) + \alpha_{18}(\boldsymbol{E}\otimes\boldsymbol{RE} - \boldsymbol{RE}\otimes\boldsymbol{E}) + \alpha_{19}\big(\boldsymbol{DR}^2 - \boldsymbol{R}^2\boldsymbol{D}\big) + \\
&+ \alpha_{20}\big(\boldsymbol{E}\otimes\boldsymbol{D}^2\boldsymbol{E} - \boldsymbol{D}^2\boldsymbol{E}\otimes\boldsymbol{E}\big) + \alpha_{21}\big(\boldsymbol{E}\otimes\boldsymbol{R}^2\boldsymbol{E} - \boldsymbol{R}^2\boldsymbol{E}\otimes\boldsymbol{E}\big) + \alpha_{22}\big(\boldsymbol{DE}\otimes\boldsymbol{D}^2\boldsymbol{E} - \boldsymbol{D}^2\boldsymbol{E}\otimes\boldsymbol{DE}\big)
\end{aligned}$$

where $\alpha_i, i = 1,...22$ may be functions of the invariants

$$\theta, |\boldsymbol{E}|^2, \mathrm{tr}\,\boldsymbol{D}^2, \mathrm{tr}\,\boldsymbol{D}^3, \mathrm{tr}\,\boldsymbol{R}, \mathrm{tr}(\boldsymbol{DE}\otimes\boldsymbol{E}), \mathrm{tr}\big(\boldsymbol{D}^2\boldsymbol{E}\otimes\boldsymbol{E}\big), \mathrm{tr}\big(\boldsymbol{E}\cdot\boldsymbol{R}^2\boldsymbol{E}\big),$$

$$\mathrm{tr}\big(\boldsymbol{DR}^2\big), \mathrm{tr}\big(\boldsymbol{D}^2\boldsymbol{R}^2\big), \mathrm{tr}\big(\boldsymbol{D}^2\boldsymbol{R}^2\boldsymbol{DR}\big), \mathrm{tr}(\boldsymbol{E}\cdot\boldsymbol{DRE}), \mathrm{tr}\big(\boldsymbol{E}\cdot\boldsymbol{D}^2\boldsymbol{RE}\big), \mathrm{tr}\big(\boldsymbol{E}\cdot\boldsymbol{RDR}^2\boldsymbol{E}\big).$$

Note that the terms with $\alpha_2 - \alpha_{14}$ are generating the symmetric part of ***S***, while the terms with $\alpha_{15} - \alpha_{22}$ are generating the skewsymmetric part of ***S***.

On the one hand, the couple stress $\boldsymbol{N}^D$ shall improve the description of the material behaviour with respect to the directional dependence on the electrical field and hence should depend on $\boldsymbol{E}$. On the other hand, it should be as simple as possible and in accordance with the classical theories. Thus, we assume that $\boldsymbol{N}^D$ is of the form

$$\boldsymbol{N}^D = \boldsymbol{N}^D\left(\theta, (\nabla \mathrm{w})_S, (\nabla \mathrm{w})_A, \boldsymbol{E}\right).$$

From representation theorems (cf. appendix of Eringen and Maugin, (1989) and references therein) it follows that the most general form for $\boldsymbol{N}^D$ is given by

$$\begin{aligned}
\boldsymbol{N}^D &= \beta_1 \boldsymbol{I} + \beta_2 \boldsymbol{E}\otimes\boldsymbol{E} + \beta_3 (\nabla \mathrm{w})_S + \beta_4 (\nabla \mathrm{w})_S^2 + \beta_5\left((\nabla \mathrm{w})_S \boldsymbol{E}\otimes\boldsymbol{E} + \boldsymbol{E}\otimes(\nabla \mathrm{w})_S \boldsymbol{E}\right) + \\
&+ \beta_6\left((\nabla \mathrm{w})_S^2 \boldsymbol{E}\otimes\boldsymbol{E} + \boldsymbol{E}\otimes(\nabla \mathrm{w})_S^2 \boldsymbol{E}\right) + \beta_7 (\nabla \mathrm{w})_A^2 + \\
&+ \beta_8\left((\nabla \mathrm{w})_S (\nabla \mathrm{w})_A - (\nabla \mathrm{w})_A (\nabla \mathrm{w})_S\right) + \beta_9\left((\nabla \mathrm{w})_A \boldsymbol{E}\otimes\boldsymbol{E} + \boldsymbol{E}\otimes(\nabla \mathrm{w})_A \boldsymbol{E}\right) + \\
&+ \beta_{10} (\nabla \mathrm{w})_A \boldsymbol{E}\otimes(\nabla \mathrm{w})_A \boldsymbol{E} + \beta_{11} (\nabla \mathrm{w})_A (\nabla \mathrm{w})_S (\nabla \mathrm{w})_A + \\
&+ \beta_{12}\left((\nabla \mathrm{w})_S^2 (\nabla \mathrm{w})_A - (\nabla \mathrm{w})_A (\nabla \mathrm{w})_S^2\right) + \\
&+ \beta_{13}\left((\nabla \mathrm{w})_A (\nabla \mathrm{w})_S (\nabla \mathrm{w})_A^2 - (\nabla \mathrm{w})_A^2 (\nabla \mathrm{w})_S (\nabla \mathrm{w})_A\right) + \\
&+ \beta_{14}\left((\nabla \mathrm{w})_A \boldsymbol{E}\otimes(\nabla \mathrm{w})_A^2 \boldsymbol{E} + (\nabla \mathrm{w})_A^2 \boldsymbol{E}\otimes(\nabla \mathrm{w})_A \boldsymbol{E}\right) + \beta_{15} (\nabla \mathrm{w})_A + \\
&+ \beta_{16}\left((\nabla \mathrm{w})_S (\nabla \mathrm{w})_A + (\nabla \mathrm{w})_A (\nabla \mathrm{w})_S\right) + \beta_{17}\left(\boldsymbol{E}\otimes(\nabla \mathrm{w})_S \boldsymbol{E} - (\nabla \mathrm{w})_S \boldsymbol{E}\otimes\boldsymbol{E}\right) + \\
&+ \beta_{18}\left(\boldsymbol{E}\otimes(\nabla \mathrm{w})_A \boldsymbol{E} - (\nabla \mathrm{w})_A \boldsymbol{E}\otimes\boldsymbol{E}\right) + \beta_{19}\left((\nabla \mathrm{w})_S (\nabla \mathrm{w})_A^2 - (\nabla \mathrm{w})_A^2 (\nabla \mathrm{w})_S\right) + \\
&+ \beta_{20}\left(\boldsymbol{E}\otimes(\nabla \mathrm{w})_S^2 \boldsymbol{E} - (\nabla \mathrm{w})_S^2 \boldsymbol{E}\otimes\boldsymbol{E}\right) + \beta_{21}\left(\boldsymbol{E}\otimes(\nabla \mathrm{w})_A^2 \boldsymbol{E} - (\nabla \mathrm{w})_A^2 \boldsymbol{E}\otimes\boldsymbol{E}\right) + \\
&+ \beta_{22}\left((\nabla \mathrm{w})_S \boldsymbol{E}\otimes(\nabla \mathrm{w})_S^2 \boldsymbol{E} + (\nabla \mathrm{w})_S^2 \boldsymbol{E}\otimes(\nabla \mathrm{w})_S \boldsymbol{E}\right)
\end{aligned}$$

where $\beta_i,\ i = 1,...22$ may be functions of the invariants

$$\begin{aligned}
&\theta, |\boldsymbol{E}|^2, \operatorname{tr}(\nabla \mathrm{w})_S, \operatorname{tr}(\nabla \mathrm{w})_S^2, \operatorname{tr}(\nabla \mathrm{w})_S^3, (\nabla \mathrm{w})_A^2, \operatorname{tr}\left((\nabla \mathrm{w})_S \boldsymbol{E}\otimes\boldsymbol{E}\right), \\
&\operatorname{tr}\left((\nabla \mathrm{w})_S^2 \boldsymbol{E}\otimes\boldsymbol{E}\right), \operatorname{tr}\left(\boldsymbol{E}\cdot(\nabla \mathrm{w})_A^2 \boldsymbol{E}\right), \operatorname{tr}\left((\nabla \mathrm{w})_S (\nabla \mathrm{w})_A^2\right), \operatorname{tr}\left((\nabla \mathrm{w})_S^2 (\nabla \mathrm{w})_A^2\right), \\
&\operatorname{tr}\left((\nabla \mathrm{w})_S^2 (\nabla \mathrm{w})_A^2 (\nabla \mathrm{w})_S (\nabla \mathrm{w})_A\right), \operatorname{tr}\left(\boldsymbol{E}\cdot(\nabla \mathrm{w})_S (\nabla \mathrm{w})_A \boldsymbol{E}\right), \\
&\operatorname{tr}\left(\boldsymbol{E}\cdot(\nabla \mathrm{w})_S^2 (\nabla \mathrm{w})_A \boldsymbol{E}\right), \operatorname{tr}\left(\boldsymbol{E}\cdot(\nabla \mathrm{w})_A (\nabla \mathrm{w})_S (\nabla \mathrm{w})_A^2 \boldsymbol{E}\right),
\end{aligned}$$

and $(\ )_S$ and $(\ )_A$ means the symmetric and skewsymmetric part, respectively. The terms with $\beta_2 - \beta_{14}$ are generating the symmetric part of $\boldsymbol{N}^D$, while the terms with $\beta_{15} - \beta_{22}$ are generating the skewsymmetric part of $\boldsymbol{N}^D$.

It would be futile to experimentally determine all these material functions and thus we are left with the task of simplifying the expressions for the stresses without forsaking the possibility to obtain a model that can reflect the behaviour of electrorheological fluids. This section is devoted to a discussion of special constitutive models with a view towards developing a theoretical framework that is amenable to analysis. Firstly, we may assume that the extra stress $\boldsymbol{S}$ is linear in $\boldsymbol{D}$ and $\boldsymbol{R}$ and quadratic in $\boldsymbol{E}$, and the couple stress

$\boldsymbol{N}^D$ is linear in $(\nabla \mathrm{w})_S, (\nabla \mathrm{w})_A$ and quadratic in $\boldsymbol{E}$. Then we obtain restrictions on the form of $\boldsymbol{S}$ and $\boldsymbol{N}^D$, which are posed by the reduced entropy inequality Eq.(2.30).

Assuming now that $\boldsymbol{S}$ is linear in $\boldsymbol{D}$ and $\boldsymbol{R}$ and has quadratic growth in $\boldsymbol{E}$, we get

$$\alpha_2 = \overline{\alpha}_1, \quad \alpha_3 = \overline{\alpha}_2 + \overline{\alpha}_3 \left|\boldsymbol{E}^2\right|, \quad \alpha_5 = \overline{\alpha}_4,$$

$$\alpha_9 = \overline{\alpha}_5 \quad \alpha_{15} = \overline{\alpha}_6 + \overline{\alpha}_7 \left|\boldsymbol{E}^2\right|, \quad \alpha_{17} = \overline{\alpha}_8, \quad \alpha_{18} = \overline{\alpha}_9,$$

$$\alpha_4 = \alpha_6 = \alpha_7 = \alpha_8 = \alpha_{10} = \alpha_{11} = \alpha_{12} = \alpha_{13} = \alpha_{14} = 0,$$

$$\alpha_{16} = \alpha_{19} = \alpha_{20} = \alpha_{21} = \alpha_{22} = 0$$

where $\overline{\alpha}_1, \ldots, \overline{\alpha}_9$ are functions of $\theta$ only. Similarly, we obtain from the assumption that $\boldsymbol{N}^D$ is linear in $(\nabla \mathrm{w})_A$ and $(\nabla \mathrm{w})_S$

$$\beta_1 = \overline{\beta}_1 + \overline{\beta}_2 |\boldsymbol{E}|^2 + \overline{\beta}_3 \mathrm{tr}(\nabla \mathrm{w})_S + \overline{\beta}_4 \boldsymbol{E} \cdot (\nabla \mathrm{w})_S \boldsymbol{E} + \overline{\beta}_5 |\boldsymbol{E}|^2 \mathrm{tr}(\nabla \mathrm{w})_S,$$

$$\beta_2 = \overline{\beta}_6 + \overline{\beta}_7 \mathrm{tr}(\nabla \mathrm{w})_S, \quad \beta_3 = \overline{\beta}_8 + \overline{\beta}_9 |\boldsymbol{E}|^2, \quad \beta_5 = \overline{\beta}_{10},$$

$$\beta_9 = \overline{\beta}_{11}, \quad \beta_{15} = \overline{\beta}_{12} + \overline{\beta}_{13} |\boldsymbol{E}|^2, \quad \beta_{17} = \overline{\beta}_{14}, \quad \beta_{18} = \overline{\beta}_{15},$$

$$\beta_4 = \beta_6 = \beta_7 = \beta_8 = \beta_{10} = \beta_{11} = \beta_{12} = \beta_{13} = \beta_{14} = 0,$$

$$\beta_{16} = \beta_{19} = \beta_{20} = \beta_{21} = \beta_{22} = 0$$

where $\overline{\beta}_1, \ldots, \overline{\beta}_{15}$ are functions of $\theta$ only. Now, holding the temperature fixed in the Clausius-Duhem inequality (3.22), we obtain

$$\begin{aligned}
&\overline{\alpha}_1 \boldsymbol{E} \cdot \boldsymbol{DE} + \left(\overline{\alpha}_2 + \overline{\alpha}_3 |\boldsymbol{E}|^2\right) |\boldsymbol{D}|^2 + 2\overline{\alpha}_4 |\boldsymbol{DE}|^2 + 2(\overline{\alpha}_5 - \overline{\alpha}_8) \boldsymbol{DE} \cdot \boldsymbol{RE} + \\
&+ \left(\overline{\alpha}_6 + \overline{\alpha}_7 |\boldsymbol{E}|^2\right) |\boldsymbol{R}|^2 - 2\overline{\alpha}_9 |\boldsymbol{RE}|^2 + \left(\overline{\beta}_1 + \overline{\beta}_2 |\boldsymbol{E}|^2\right) \mathrm{tr}(\nabla \mathrm{w})_S + \\
&+ \left(\overline{\beta}_3 + \overline{\beta}_5 |\boldsymbol{E}|^2\right) \left|\mathrm{tr}(\nabla \mathrm{w})_S\right|^2 + (\overline{\beta}_4 + \overline{\beta}_7) \mathrm{tr}(\nabla \mathrm{w})_S \boldsymbol{E} \cdot (\nabla \mathrm{w})_S \boldsymbol{E} + \\
&+ \overline{\beta}_6 \boldsymbol{E} \cdot (\nabla \mathrm{w})_S \boldsymbol{E} + \left(\overline{\beta}_8 + \overline{\beta}_9 |\boldsymbol{E}|^2\right) \left|(\nabla \mathrm{w})_S\right|^2 + 2\overline{\beta}_{10} \left|(\nabla \mathrm{w})_S \boldsymbol{E}\right|^2 + \\
&+ 2(\overline{\beta}_{11} - \overline{\beta}_{14}) (\nabla \mathrm{w})_S \boldsymbol{E} \cdot (\nabla \mathrm{w})_A \boldsymbol{E} + \left(\overline{\beta}_{12} + \overline{\beta}_{13} |\boldsymbol{E}|^2\right) \left|(\nabla \mathrm{w})_A\right|^2 + \\
&- 2\overline{\beta}_{15} \left|(\nabla \mathrm{w})_A \boldsymbol{E}\right|^2 \geq 0.
\end{aligned} \tag{4.1}$$

This inequality has to hold for all $\boldsymbol{D}$, $\boldsymbol{R}$, $(\nabla \mathrm{w})_S$, $(\nabla \mathrm{w})_A$ and $\boldsymbol{E}$. By specifying and rescaling their values we obtain restrictions on the remaining material parameters[8].

[8] A similar rescaling argument was used in a completely different context by Nečas and Šilhavý (1991), Růžička (1992) and also in Rajagopal and Růžička (1996), Růžička (2000).

Inequality (4.1) splits into 2 inequalities, one including all terms with $\boldsymbol{D}$ and $\boldsymbol{R}$ and the other including all terms with $(\nabla \mathrm{w})_S$ and $(\nabla \mathrm{w})_A$, which will be discussed separately.

*Firstly*, we set $\nabla \mathrm{w} = \boldsymbol{0}$. In the remaining part of Eq.(4.1) we first choose $\boldsymbol{R} = \boldsymbol{0}$ and thus Eq.(4.1) reads

$$\overline{\alpha}_1 \boldsymbol{E} \cdot \boldsymbol{DE} + \left(\overline{\alpha}_2 + \overline{\alpha}_3 |\boldsymbol{E}|^2\right) |\boldsymbol{D}|^2 + 2\overline{\alpha}_4 |\boldsymbol{DE}|^2 \geq 0 .$$

Thus we are in the same situation as in the case of a linear incompressible non-polar electrorheological fluid and we can proceed in the same way [cf. (Růžička, 2000), Lemma 1.3.34].

Setting $\boldsymbol{E} = \boldsymbol{0}$ we get

$$\overline{\alpha}_2 \geq 0 . \tag{4.2}$$

If we replace $\boldsymbol{D}$ by $\gamma\boldsymbol{D}$, multiply by $\gamma^{-1}$ and let $\gamma \to 0$ we obtain

$$\overline{\alpha}_1 = 0 .$$

Setting $\boldsymbol{DE} = 0$, rescaling $\boldsymbol{E} \to \gamma\boldsymbol{E}$, multiplying by $\gamma^{-2}$, letting $\gamma \to \infty$ then yields

$$\overline{\alpha}_3 \geq 0 , \tag{4.3}$$

and (rescaling $\boldsymbol{E} \to \gamma\boldsymbol{E}$, multiplying by $\gamma^{-2}$, letting $\gamma \to \infty$ )

$$\overline{\alpha}_3 |\boldsymbol{E}|^2 |\boldsymbol{D}|^2 + 2\overline{\alpha}_4 |\boldsymbol{DE}|^2 \geq 0 .$$

Using in this inequality that $|\boldsymbol{DE}|^2 \leq \frac{2}{3} |\boldsymbol{D}|^2 |\boldsymbol{E}|^2$, where equality is attained, [cf. Lemma 1.3.7 (Růžička, 2000)] we deduce

$$\overline{\alpha}_3 + \frac{4}{3}\overline{\alpha}_4 \geq 0 . \tag{4.4}$$

Now setting $\boldsymbol{D} = \boldsymbol{0}$ and $\nabla \mathrm{w} = \boldsymbol{0}$ in Eq.(4.1) we have

$$\left(\overline{\alpha}_6 + \overline{\alpha}_7 |\boldsymbol{E}|^2\right) |\boldsymbol{R}|^2 - 2\overline{\alpha}_9 |\boldsymbol{RE}|^2 \geq 0 ,$$

and we deduce in the same way as above

$$\overline{\alpha}_6 \geq 0 , \qquad \overline{\alpha}_7 \geq 0 , \qquad \overline{\alpha}_7 \geq \overline{\alpha}_9 \tag{4.5}$$

where we used $|\boldsymbol{RE}|^2 \leq \frac{1}{2} |\boldsymbol{R}|^2 |\boldsymbol{E}|^2$ for the last inequality. Finally, we obtain by rescaling $\boldsymbol{E} \to \gamma\boldsymbol{E}$, multiplying by $\gamma^{-2}$, letting $\gamma \to \infty$, and by changing the sign of $\boldsymbol{D}$ that

$$\overline{\alpha}_3|\boldsymbol{E}|^2|\boldsymbol{D}|^2+2\overline{\alpha}_4|\boldsymbol{DE}|^2-2|\overline{\alpha}_5-\overline{\alpha}_8|\,|\boldsymbol{DE}\cdot\boldsymbol{RE}|+\overline{\alpha}_7|\boldsymbol{E}|^2|\boldsymbol{R}|^2-2\overline{\alpha}_9|\boldsymbol{RE}|^2\geq 0\,,$$

from which we deduce, on choosing $\boldsymbol{D}$, $\boldsymbol{R}$, $\boldsymbol{E}$ such that $|\boldsymbol{DE}|^2=\frac{2}{3}|\boldsymbol{D}|^2|\boldsymbol{E}|^2$, $|\boldsymbol{RE}|^2=\frac{1}{2}|\boldsymbol{R}|^2|\boldsymbol{E}|^2$ and $|\boldsymbol{DE}|^2=\frac{1}{2}|\boldsymbol{D}|^2|\boldsymbol{E}|^2$, $|\boldsymbol{RE}|^2=\frac{1}{2}|\boldsymbol{R}|^2|\boldsymbol{E}|^2$, $|\boldsymbol{DE}\cdot\boldsymbol{RE}|=\frac{1}{2}|\boldsymbol{D}||\boldsymbol{R}||\boldsymbol{E}|^2$, respectively

$$\overline{\alpha}_9\leq\overline{\alpha}_3+\frac{4}{3}\overline{\alpha}_4+\overline{\alpha}_7\,,$$

$$\overline{\alpha}_9+|\overline{\alpha}_5-\overline{\alpha}_8|\leq\overline{\alpha}_3+\overline{\alpha}_4+\overline{\alpha}_7\,, \tag{4.6}$$

$$|\overline{\alpha}_5-\overline{\alpha}_8|^2\leq 4(\overline{\alpha}_3+\overline{\alpha}_4)(\overline{\alpha}_7-\overline{\alpha}_9)\,.$$

*Secondly*, we choose $\boldsymbol{D}=\boldsymbol{R}=\boldsymbol{0}$. Setting now in the remaining inequality (4.1) $(\nabla \mathrm{w})_A=\boldsymbol{0}$, we are in a very similar situation as in the case of a linear compressible non-polar electrorheological fluid [cf. (Růžička, 2000), sec. 1.3.1]. By setting $\boldsymbol{E}=\boldsymbol{0}$ we get

$$\overline{\beta}_1\,tr(\nabla \mathrm{w})_S+\overline{\beta}_3|tr(\nabla \mathrm{w})_S|^2+\overline{\beta}_8|(\nabla \mathrm{w})_S|^2\geq 0\,,$$

from which we deduce by an appropriate rescaling of $(\nabla \mathrm{w})_S$, choosing $tr(\nabla \mathrm{w})_S=0$ and by choosing $\boldsymbol{D}=\boldsymbol{I}$, respectively,

$$\overline{\beta}_1=0\,,\qquad \overline{\beta}_8\geq 0\,,\qquad 3\overline{\beta}_3+\overline{\beta}_8\geq 0\,. \tag{4.7}$$

Rescaling now $\boldsymbol{E}\to\gamma\boldsymbol{E}$, multiplying by $\gamma^{-2}$, letting $\gamma\to\infty$, we obtain from Eq.(4.1)

$$\begin{aligned}&\overline{\beta}_2|\boldsymbol{E}|^2\,\mathrm{tr}(\nabla \mathrm{w})_S+(\overline{\beta}_4+\overline{\beta}_7)\boldsymbol{E}\cdot(\nabla \mathrm{w})_S\,\boldsymbol{E}\,\mathrm{tr}(\nabla \mathrm{w})_S+\overline{\beta}_5|\boldsymbol{E}|^2|\mathrm{tr}(\nabla \mathrm{w})_S|^2+\\&+\overline{\beta}_6\boldsymbol{E}\cdot(\nabla \mathrm{w})_S\,\boldsymbol{E}+\overline{\beta}_9|\boldsymbol{E}|^2|(\nabla \mathrm{w})_S|^2+2\overline{\beta}_{10}|(\nabla \mathrm{w})_S\,\boldsymbol{E}|^2\geq 0.\end{aligned} \tag{4.8}$$

One sees that the coeffcients in front of the linear terms in $(\nabla \mathrm{w})_S$ have to vanish, i.e.,

$$\overline{\beta}_2=\overline{\beta}_6=0\,.$$

Choosing $tr(\nabla \mathrm{w})_S=0$ we get (cf. Eq.(4.4))

$$\overline{\beta}_9\geq 0\,,\qquad \overline{\beta}_9+\frac{4}{3}\overline{\beta}_{10}\geq 0\,, \tag{4.9}$$

while $(\nabla \mathrm{w})_S\,\boldsymbol{E}=\boldsymbol{0}$ implies

$$3\overline{\beta}_5+\overline{\beta}_9\geq 0\,. \tag{4.10}$$

Now we are exactly in the same situation as for a linear compressible non-polar electrorheological fluid. Thus we decompose $(\nabla \mathrm{w})_S$ as

$$(\nabla \mathrm{w})_S = \frac{1}{3}\left(\mathrm{tr}(\nabla \mathrm{w})_S\right)\boldsymbol{I} + \boldsymbol{G}, \qquad \mathrm{tr}\,\boldsymbol{G} = 0$$

where now $\boldsymbol{G}$ and $\mathrm{tr}\,(\nabla \mathrm{w})_S$ may be chosen independently. Thus inequality (4.8) can be re-written as

$$\begin{gathered}\left|tr\,(\nabla \mathrm{w})_S\right|^2 |\boldsymbol{E}|^2 \left\{\overline{\beta}_5 + \frac{1}{3}\overline{\beta}_9 + \frac{1}{3}\overline{\beta}_4 + \frac{1}{3}\overline{\beta}_7 + \frac{2}{9}\overline{\beta}_{10}\right\} + \\ + \left|tr\,(\nabla \mathrm{w})_S\right| \left|\boldsymbol{E}\cdot\boldsymbol{GE}\right| \left|\overline{\beta}_4 + \overline{\beta}_7 + \frac{4}{3}\overline{\beta}_{10}\right| + \overline{\beta}_9 |\boldsymbol{E}|^2 |\boldsymbol{G}|^2 + 2\overline{\beta}_{10} |\boldsymbol{GE}|^2 \geq 0\end{gathered} \tag{4.11}$$

where we also changed the sign of $\boldsymbol{G}$. Choosing now $\boldsymbol{G} = 0$ provides

$$\overline{\beta}_5 + \frac{1}{3}\left(\overline{\beta}_4 + \overline{\beta}_7 + \overline{\beta}_9 + \frac{2}{3}\overline{\beta}_{10}\right) \geq 0\,. \tag{4.12}$$

The right-hand side of Eq.(4.11) is a polynomial of second order in $\left|tr\,(\nabla \mathrm{w})_S\right|$ and its non-negativity is equivalent to the condition

$$\begin{gathered}\left|\overline{\beta}_7 + \overline{\beta}_4 + \frac{4}{3}\overline{\beta}_{10}\right|^2 + |\boldsymbol{E}\cdot\boldsymbol{GE}|^2 \leq 4|\boldsymbol{E}|^2 \left(\overline{\beta}_5 + \frac{1}{3}\overline{\beta}_9 + \right. \\ \left. + \frac{1}{3}\overline{\beta}_4 + \frac{1}{3}\overline{\beta}_7 + \frac{2}{9}\overline{\beta}_{10}\right)\left(\overline{\beta}_9 |\boldsymbol{E}|^2 |\boldsymbol{G}|^2 + 2\overline{\beta}_{10} |\boldsymbol{GE}|^2\right),\end{gathered} \tag{4.13}$$

from which we deduce using $|\boldsymbol{GE}|^2 \leq \frac{2}{3}|\boldsymbol{E}|^2|\boldsymbol{G}|^2$ and $|\boldsymbol{E}\cdot\boldsymbol{GE}|^2 \leq \frac{2}{3}|\boldsymbol{E}|^2|\boldsymbol{G}|$ where equality is attained in both inequalities for the same choice of $\boldsymbol{E}$ and $\boldsymbol{G}$ [cf. (Růžička, 2000), Lemma 1.3.28]

$$\left|\overline{\beta}_7 + \overline{\beta}_4 + \frac{4}{3}\overline{\beta}_{10}\right|^2 \leq 6\left(\overline{\beta}_5 + \frac{1}{3}\left(\overline{\beta}_4 + \overline{\beta}_7 + \overline{\beta}_9 + \frac{2}{3}\overline{\beta}_{10}\right)\right)\left(\overline{\beta}_9 + \frac{4}{3}\overline{\beta}_{10}\right). \tag{4.14}$$

Setting now in Eq.(4.1) $\boldsymbol{D} = \boldsymbol{R} = (\nabla \mathrm{w})_S = \boldsymbol{0}$ and varying $(\nabla \mathrm{w})_A$, one immediately deduces (cf. (4.5))

$$\overline{\beta}_{12} \geq 0\,, \qquad \overline{\beta}_{13} \geq 0\,, \qquad \overline{\beta}_{13} \geq \overline{\beta}_{15}\,. \tag{4.15}$$

Decomposing again $(\nabla \mathrm{w})_S = \frac{1}{3}\left(\mathrm{tr}(\nabla \mathrm{w})_S\right) + \boldsymbol{G}$, we can vary $\mathrm{tr}(\nabla \mathrm{w})_S$ and $\boldsymbol{G}$ independently, and derive from Eq.(4.1) with $\boldsymbol{D} = \boldsymbol{R} = \boldsymbol{0}$, after rescaling $\boldsymbol{E}$ and on changing the sign of $\boldsymbol{G}$ [cf. (4.11)]

$$
\begin{aligned}
&\left|\operatorname{tr}(\nabla \mathrm{w})_S\right|^2 |\boldsymbol{E}|^2\left[\overline{\beta}_5+\frac{1}{3}\left(\overline{\beta}_9+\overline{\beta}_4+\overline{\beta}_7\right)+\frac{2}{9}\overline{\beta}_{10}\right]+\\
&-\left|\operatorname{tr}(\nabla \mathrm{w})_S\right|\left|\left(\overline{\beta}_4+\overline{\beta}_7\right)+\frac{4}{3}\overline{\beta}_{10}\right|\left|\boldsymbol{E}\cdot\boldsymbol{GE}\right|+\\
&+\overline{\beta}_9|\boldsymbol{E}|^2|\boldsymbol{G}|^2+2\overline{\beta}_{10}|\boldsymbol{GE}|^2-2\left|\overline{\beta}_{11}-\overline{\beta}_{14}\right|\left|\boldsymbol{GE}\cdot(\nabla \mathrm{w})_A\boldsymbol{E}\right|+\\
&+\overline{\beta}_{13}|\boldsymbol{E}|^2\left|(\nabla \mathrm{w})_A\right|^2-2\overline{\beta}_{15}\left|(\nabla \mathrm{w})_A\boldsymbol{E}\right|^2\geq 0.
\end{aligned}
\tag{4.16}
$$

The choice $\operatorname{tr}(\nabla \mathrm{w})_S=0$ yields the analogue of Eq.(4.6), namely

$$
\begin{aligned}
&\overline{\beta}_{15}\leq\overline{\beta}_9+\frac{4}{3}\overline{\beta}_{10}+\overline{\beta}_{13},\\
&\overline{\beta}_{15}+\left|\overline{\beta}_{11}-\overline{\beta}_{14}\right|\leq\overline{\beta}_9+\overline{\beta}_{10}+\overline{\beta}_{13},\\
&\left|\overline{\beta}_{11}-\overline{\beta}_{14}\right|^2\leq 4\left(\overline{\beta}_9+\overline{\beta}_{10}\right)\left(\overline{\beta}_{13}-\overline{\beta}_{15}\right).
\end{aligned}
\tag{4.17}
$$

Inequality Eq.(4.16) is quadratic in $\left|\operatorname{tr}(\nabla \mathrm{w})_S\right|$ and thus we get [cf. (4.13)]

$$
\begin{aligned}
&\left|\overline{\beta}_4+\overline{\beta}_7+\frac{4}{3}\overline{\beta}_{10}\right|^2+\left|\boldsymbol{E}\cdot\boldsymbol{GE}\right|^2\leq 4\Bigg(\overline{\beta}_5+\\
&+\frac{1}{3}\left(\overline{\beta}_9+\overline{\beta}_4+\overline{\beta}_7\right)+\frac{2}{9}\overline{\beta}_{10}\Bigg)\Big(\overline{\beta}_9|\boldsymbol{E}|^2|\boldsymbol{G}|^2+2\overline{\beta}_{10}|\boldsymbol{GE}|^2+\\
&-2\left|\overline{\beta}_{11}-\overline{\beta}_{14}\right|\left|\boldsymbol{GE}\cdot(\nabla \mathrm{w})_A\boldsymbol{E}\right|+\overline{\beta}_{13}|\boldsymbol{E}|^2\left|(\nabla \mathrm{w})_A\right|^2-2\overline{\beta}_{15}\left|(\nabla \mathrm{w})_A\boldsymbol{E}\right|^2\Big),
\end{aligned}
\tag{4.18}
$$

from which one can deduce [cf. (4.6)]

$$
\begin{aligned}
&\left|\overline{\beta}_4+\overline{\beta}_7+\frac{4}{3}\overline{\beta}_{10}\right|^2\leq 6\left(\overline{\beta}_5+\frac{1}{3}\left(\overline{\beta}_9+\overline{\beta}_4+\overline{\beta}_7\right)+\frac{2}{9}\overline{\beta}_{10}\right)\left(\overline{\beta}_9+\frac{4}{3}\overline{\beta}_{10}+\overline{\beta}_{13}-\overline{\beta}_{15}\right),\\
&\left|\overline{\beta}_4+\overline{\beta}_7+\frac{4}{3}\overline{\beta}_{10}\right|^2\leq 8\left(\overline{\beta}_5+\frac{1}{3}\left(\overline{\beta}_9+\overline{\beta}_4+\overline{\beta}_7\right)+\frac{2}{9}\overline{\beta}_{10}\right)\left(2\overline{\beta}_9+2\overline{\beta}_{10}-\left|\overline{\beta}_{11}-\overline{\beta}_{14}\right|+\overline{\beta}_{13}-\overline{\beta}_{15}\right).
\end{aligned}
\tag{4.19}
$$

Furthermore, we require the extra stress to be symmetric, if $\boldsymbol{E}=\boldsymbol{0}$, thus choosing $\overline{\alpha}_6=0$. For the couple stress we choose $\overline{\beta}_3=0$. Note that if either $\overline{\beta}_8>0$ or $\overline{\beta}_{12}>0$ the couple stress does not vanish, if $\boldsymbol{E}=\boldsymbol{0}$.

Summarizing, the constitutive equations for the extra stress and the couple stress, considered in the remainder of this paper, are

$$
\begin{aligned}
\boldsymbol{S}=&\left(\overline{\alpha}_2+\overline{\alpha}_2|\boldsymbol{E}|^2\right)\boldsymbol{D}+\overline{\alpha}_4(\boldsymbol{DE}\otimes\boldsymbol{E}+\boldsymbol{E}\otimes\boldsymbol{DE})+\overline{\alpha}_5(\boldsymbol{RE}\otimes\boldsymbol{E}+\boldsymbol{E}\otimes\boldsymbol{RE})+\\
&+\overline{\alpha}_7|\boldsymbol{E}|^2\boldsymbol{R}+\overline{\alpha}_8(\boldsymbol{E}\otimes\boldsymbol{DE}-\boldsymbol{DE}\otimes\boldsymbol{E})+\overline{\alpha}_9(\boldsymbol{E}\otimes\boldsymbol{RE}-\boldsymbol{RE}\otimes\boldsymbol{E}),
\end{aligned}
\tag{4.20}
$$

$$\begin{aligned} N^D = &\left(\overline{\beta}_4 E\cdot(\nabla w)_S E + \overline{\beta}_5 |E|^2 \operatorname{tr}(\nabla w)_S\right) I + \overline{\beta}_7 \operatorname{tr}(\nabla w)_S E\otimes E + \\ &+\left(\overline{\beta}_8 + \overline{\beta}_9 |E|^2\right)(\nabla w)_S + \overline{\beta}_{10}\left((\nabla w)_S E\otimes E + E\otimes(\nabla w)_S E\right) + \\ &+\overline{\beta}_{11}\left((\nabla w)_A E\otimes E + E\otimes(\nabla w)_A E\right) + \left(\overline{\beta}_{12} + \overline{\beta}_{13}|E|^2\right)(\nabla w)_A + \\ &+\overline{\beta}_{14}\left(E\otimes(\nabla w)_S E - (\nabla w)_S E\otimes E\right) + \overline{\beta}_{15}\left(E\otimes(\nabla w)_A E - (\nabla w)_A E\otimes E\right). \end{aligned} \tag{4.21}$$

Note that $\overline{\alpha}_2 - \overline{\alpha}_5$, $\overline{\alpha}_7 - \overline{\alpha}_9$ and $\overline{\beta}_4, \overline{\beta}_5, \overline{\beta}_7 - \overline{\beta}_{15}$ are functions of $\theta$ only. In the following sections we assume that all nontrivial inequalities in Eqs (4.2), (4.3)-(4.7), (4.9), (4.10), (4.12), (4.14), (4.15) and (4.17)-(4.19) are strict. These constitutive equations look still complicated, but we do not want to simplify them before the important case of a viscometric flow is studied.

The system (3.17)-(3.21), with $S$ and $N^D$ given above, and Eqs (3.10)-(3.13) completed with appropriate boundary and initial conditions describes the behaviour of a micropolar ERF.

## 5. Viscometric flow

To show the behaviour of the velocity and the importance of the material parameters, a *viscometric flow* is studied in this section. We determine analytical solutions in a pressure driven channel flow and a simple shear flow and illustrate selected results.

It is our purpose to show that the micropolar theory may be used for electrorheological fluids if a dependence on the direction of the electric field is to be modelled.

It is not our intention to review or discuss general results of micropolar theories nor other properties of electrorheological fluids that have been observed.

Before we start the calculations, it is necessary to specify the model that we want to use. Indeed, there are „different" micropolar fluids, depending on the choice of the micro-inertia tensor which represents symmetries of the microstructures in the fluid. In Eringen (1966) and Stokes (1984), *isotropic micropolar* fluids are investigated, while Eringen studied *anisotropic micropolar* fluids in Eringen (1980). In the former case, the micro-inertia tensor has a diagonal form and only one (constant) component $Q = \Theta I$, while in the latter case it has in general six components differing from each other. For the sake of simplicity we are dealing here with isotropic micropolar fluids. Note that in this case the balance of micro-inertia (2.12) implies that $\Theta$ is materially constant.

Now let us consider steady state flow configurations also described in Stokes (1984), where Stokes studied simple shear flows, channel flows and pipe flows, respectively. The latter case was also studied in sec. 3 of Eringen (1966)[9].

Thus, we investigate a steady state flow of a micropolar ERF between two parallel infinite plates (compare Eringen (1966) and (1980)) with the following properties

$$v_2 = v_3 = 0, \qquad v_1 = v_1(x_2), \qquad \omega_1 = \omega_2 = 0, \qquad \omega_3 = \omega_3(x_2), \tag{5.1}$$

$$D_{12} = D_{21} = \frac{1}{2} v_{1,2}, \qquad D_{ij} = 0 \quad \text{otherwise}, \tag{5.2}$$

[9] Note that if we considered anisotropic micropolar fluids (i.e., rigid particles of arbitrary shape), the motion would inherently unsteady, as pointed out in Happel and Brenner (1965), p.161. However, if both the translational and rotational Reynolds numbers are small, it is permissible to adopt a quasi-static form of the equations which was also assumed in Eringen (1980). Nevertheless, we are not interested in anisotropic micropolar fluids and thus may study steady state flows without any additional restriction.

$$R_{12} = -R_{21} = \frac{1}{2}\nu_{1,2} + \omega_3\,, \qquad R_{ij} = 0 \quad \text{otherwise} \tag{5.3}$$

where we switched to index notation. The flow direction is the positive $x_1$ - direction.

Furthermore we assume the (plane) electric field to be constant with respect to space and time, i.e.,

$$E_1 = \text{const.}\,, \qquad E_2 = \text{const.}\,, \qquad E_3 = 0\,[10]. \tag{5.4}$$

For the sake of simplicity the temperature is held constant, resulting in constant material parameters. The boundary conditions are given as

$$\nu_1(x_2 = 0) = 0\,, \qquad \nu_1(x_2 = h) = V_0\,, \tag{5.5}$$

$$\omega_3(x_2 = 0) = \omega_l\,, \qquad \omega_3(x_2 = h) = \omega_u \tag{5.6}$$

where $\omega_u$ and $\omega_l$ are functions of $|\boldsymbol{E}|^2$ such that $\omega_u = 0$ and $\omega_l = 0$ if $\boldsymbol{E} = \boldsymbol{0}$. Note that the latter restriction guarantees that $\omega_3 = 0$ is a solution of Eq.(5.10) below. We have also investigated the well-known boundary condition

$$\omega_3 = -n\nu_{1,2} \quad \text{at} \quad x_2 = 0 \quad \text{and} \quad x_2 = h\,, \tag{5.7}$$

as is discussed in detail by Kirwan in (1986), for example. Although the solutions using the „Neumann conditions” (5.7) are slightly simpler than solutions with the „Dirichlet conditions” (5.6), we found that for our purposes the latter ones seem to produce „more appropriate” solutions.

From the balance of momentum (3.18) it follows, on using (5.1)-(5.4) and $\boldsymbol{f} = \boldsymbol{0}$, that

$$S_{12,2} = \pi_{,1}\,, \qquad S_{22,2} = \pi_{,2}\,. \tag{5.8}$$

Recall that it follows from the assumptions (5.1)-(5.4), that $S_{12}$ and $S_{22}$ calculated from Eq.(4.20) cannot depend on $x_1$. Now integrating both Eqs (5.8) to obtain $\pi$ and compare the resulting terms with each other, we conclude that the pressure must be given as

$$\pi = -Kx_1 + S_{22}(x_2) \tag{5.9}$$

where $K = -S_{12,2}$ is the (prescribed) pressure gradient in $x_1$ - direction in case of a pressure driven flow. Using Eqs (5.1)-(5.4) and $\boldsymbol{1} = \boldsymbol{0}$, the balance of internal spin (3.19) reduces to

$$S_{21} - S_{12} + N_{32,2} = 0\,, \tag{5.10}$$

provided that the micro-inertia tensor $\Theta_{ij}$ has a diagonal form (as was already said).

Let us now compare the shear stresses $S_{12}$ and $S_{21}$ in a viscometric flow. From Eq.(4.20) it follows, on using the simplifications, that

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[10] Surely, the electric field will be space-dependent in real applications. On the other hand, it can be shown that a viscometric flow is not possible if the electric field is space-dependent. Thus, we are forced to make this assumption, else no closed analytical solution is possible.

$$S_{12}(x_2)=\frac{1}{2}\Big[\overline{\alpha}_2+(\overline{\alpha}_3+\overline{\alpha}_4)(E_1^2+E_2^2)+\underline{\overline{\alpha}_8(E_1^2-E_2^2)}\Big]\nu_{1,2}+ \\ +\Big[\underline{(\overline{\alpha}_7-\overline{\alpha}_9)(E_1^2+E_2^2)}-\overline{\alpha}_5(E_1^2-E_2^2)\Big]\left(\frac{1}{2}\nu_{1,2}+\omega_3\right), \tag{5.11}$$

$$S_{21}(x_2)=\frac{1}{2}\Big[\overline{\alpha}_2+(\overline{\alpha}_3+\overline{\alpha}_4)(E_1^2+E_2^2)-\underline{\overline{\alpha}_8(E_1^2-E_2^2)}\Big]\nu_{1,2}+ \\ -\Big[\underline{(\overline{\alpha}_7-\overline{\alpha}_9)(E_1^2+E_2^2)}+\overline{\alpha}_5(E_1^2-E_2^2)\Big]\left(\frac{1}{2}\nu_{1,2}+\omega_3\right). \tag{5.12}$$

Now it is much clearer to see that the underlined terms are responsible for nonsymmetric shear stresses. In particular, we obtain for electric fields perpendicular (i.e., $S_{12}(E_1=0, E_2=E_0)=: S_{12}^{E_2}$ and electric fields parallel (i.e., $S_{12}(E_1=E_0, E_2=E_0)=: S_{12}^{E_1}$ to the flow direction

$$S_{12}^{E_2}=\frac{1}{2}\Big[\overline{\alpha}_2+(\overline{\alpha}_3+\overline{\alpha}_4)E_0^2-\overline{\alpha}_8E_0^2\Big]\nu_{1,2}+\Big[(\overline{\alpha}_7-\overline{\alpha}_9)E_0^2+\overline{\alpha}_5E_0^2\Big]\left(\frac{1}{2}\nu_{1,2}+\omega_3\right), \tag{5.13}$$

$$S_{12}^{E_1}=\frac{1}{2}\Big[\overline{\alpha}_2+(\overline{\alpha}_3+\overline{\alpha}_4)E_0^2+\overline{\alpha}_8E_0^2\Big]\nu_{1,2}+\Big[(\overline{\alpha}_7-\overline{\alpha}_9)E_0^2-\overline{\alpha}_5E_0^2\Big]\left(\frac{1}{2}\nu_{1,2}+\omega_3\right), \tag{5.14}$$

respectively. Clearly, they are not the same in general, even if the velocity fields were the same: directly due to the existence of $\overline{\alpha}_5$ and $\overline{\alpha}_8$ and indirectly due to $\overline{\alpha}_7-\overline{\alpha}_9$, because $\omega_3$ may not be the same in both cases.

The normal stress differences are given by

$$S_{11}-S_{22}=4\overline{\alpha}_5E_1E_2\left(\frac{1}{2}\nu_{1,2}+\omega_3\right),$$

$$S_{22}-S_{33}=\overline{\alpha}_4E_1E_2\nu_{1,2}-2\overline{\alpha}_5E_1E_2\left(\frac{1}{2}\nu_{1,2}+\omega_3\right).$$

The only relevant couple stress component $N_{32}$ is given by Eq.(4.21) as

$$N_{32}=\left[\frac{1}{2}(\overline{\beta}_8+\overline{\beta}_{12})+\frac{1}{2}(\overline{\beta}_9+\overline{\beta}_{13})(E_1^2+E_2^2)+\frac{1}{2}(\overline{\beta}_{10}+\overline{\beta}_{11}-\overline{\beta}_{14}-\overline{\beta}_{15})E_2^2\right]\omega_{3,2}. \tag{5.15}$$

From Eqs $(4.9)_1$, $(4.15)_{2,3}$ and $(4.17)_2$ it follows that the coefficient in front of $\omega_{3,2}$ in Eq.(5.15) is strictly positive. Introducing the abbreviations

$$\eta_1=\frac{1}{2}\Big[\overline{\alpha}_2+(\overline{\alpha}_3+\overline{\alpha}_4)(E_1^2+E_2^2)+\overline{\alpha}_8(E_1^2-E_2^2)+ \\ +(\overline{\alpha}_7-\overline{\alpha}_9)(E_1^2+E_2^2)-\overline{\alpha}_5(E_1^2-E_2^2)\Big],$$

$$\eta_2=(\overline{\alpha}_7-\overline{\alpha}_9)(E_1^2+E_2^2)-\overline{\alpha}_5(E_1^2-E_2^2),$$

it follows from Eqs (4.2) and $(4.6)_2$ that

$$\eta_1 > 0 .$$

Now from Eqs (5.8), (5.9) and (5.11) we deduce

$$\omega_3 = -\frac{Kx_2}{\eta_2} + \frac{C_1}{\eta_2} - \frac{\eta_1}{\eta_2}\nu_{1,2} , \qquad \eta_2 \neq 0 \text{ [11]} \tag{5.16}$$

where $C_1$ is an integration constant. Note that $\omega_3$ is linear in the shear rate $\nu_{1,2}$ which is due to the linear dependence of ***S*** on ***R*** and ***D***.

Using Eqs (5.11), (5.12) and (5.15) and inserting them in (5.10), the following equation emerges

$$-\eta_3\nu_{1,2} - \eta_4\omega_3 + \zeta_1\omega_{3,22} = 0 \tag{5.17}$$

where we have introduced the abbreviations

$$\eta_3 = \overline{\alpha}_8\left(E_1^2 - E_2^2\right) + \left(\overline{\alpha}_7 - \overline{\alpha}_9\right)\left(E_1^2 + E_2^2\right),$$

$$\eta_4 = 2\left(\overline{\alpha}_7 - \overline{\alpha}_9\right)\left(E_1^2 + E_2^2\right),$$

$$\zeta_1 = \frac{1}{2}\left(\overline{\beta}_8 + \overline{\beta}_{12}\right) + \frac{1}{2}\left[\left(\overline{\beta}_9 + \overline{\beta}_{13}\right)\left(E_1^2 + E_2^2\right) + \left(\overline{\beta}_{10} + \overline{\beta}_{11} - \overline{\beta}_{14} - \overline{\beta}_{15}\right)E_2^2\right].$$

Note that due to Eqs $(4.5)_3$, $(4.9)_1$, $(4.15)_{2,3}$ and $(4.17)_2$ it follows that

$$\eta_4 > 0 \quad \text{for} \quad \boldsymbol{E} \neq \boldsymbol{0} , \qquad \zeta_1 > 0 . \tag{5.18}$$

Eliminating $\omega_3$ in Eq.(5.17) by means of Eq.(5.16) and integrating the emerging equation w.r.t. $x_2$, a linear ordinary differential equation of second order in $\nu_1$ alone is obtained,i.e.

$$\nu_{1,22} - \frac{1}{l^2}\nu_1 = Z_1 K x_2^2 - 2Z_1C_1x_2 + C_2$$

where

$$\frac{1}{l^2} = \frac{\eta_4\eta_1 - \eta_2\eta_3}{\zeta_1\eta_1} \qquad \text{and} \qquad Z_1 = \frac{\eta_4}{2\zeta_1\eta_1} . \tag{5.19}$$

$l$ is an internal lengthscale. Note that due to physical reasons the right-hand side of $\text{Eq.}(5.19)_1$ must be positive. An appropriate particular solution of Eq.(2.5) and the general solution of the corresponding homogeneous differential equation can easily be obtained, implying

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[11] When $\eta_2 = 0, S_{12}$ is not depending on $\omega_3$; we can determine $\nu_1$ from Eq.(5.8) without knowledge of $\omega_3$. The simple parabolic profile is obtained for a pressure driven flow (and the linear profile for a simple shear flow). Once $\nu_1$ is known, $\omega_3$ can be determined from Eq.(5.10). Note that the extra stress is still not symmetric.

$$\nu_1^p := l^2\left[-Z_1K\left(2l^2+x_2^2\right)+2Z_1C_1x_2-C_2\right], \tag{5.20}$$

$$\nu_1^h := C_3\cosh x_2/l + C_4\sinh x_2/l\,. \tag{5.21}$$

Due to its linearity the general solution of Eq.(2.5) can be obtained by superposition of the particular and the homogeneous solution.

For the determination of the four unknown constants $C_1, C_2, C_3$ and $C_4$ we use the four boundary conditions $(5.5)_{1,2}$ and $(5.6)_{1,2}$. Using Eqs (5.20) and (5.21), we obtain

$$\begin{aligned}\nu_1 = &-l^2\left[Z_1K\left(2l^2+x_2^2\right)-2Z_1C_1x_2+C_2\right] + \\ &+l^2\left[Z_1K\left(2l^2+h^2\right)-2Z_1C_1h+C_2\right]\frac{\sinh(x_2/l)}{\sinh(h/l)}+ \\ &+l^2\left(C_2+2l^2Z_1K\right)\left[\cosh(x_2/l)-\sinh(x_2/l)\frac{\cosh(h/l)}{\sinh(h/l)}\right]+V_0\,\frac{\sinh(x_2/l)}{\sinh(h/l)}\end{aligned}$$

where

$$\begin{aligned}C_2 := &\left[\left(-C_1l+2\eta_1l^3Z_1C_1+\omega_{3l}\eta_2l\right)\sinh(h/l)+l^2Z_1\left(2Kl^2\eta_1-2hC_1\eta_1+Kh^2\eta_1\right) + \right. \\ &\left.+V_0\eta_1-2\eta_1l^4Z_1K\cosh(h/l)\right]\Big/\left[\eta_1l^2\left(\cosh(h/l)\right)-1\right],\end{aligned} \tag{5.23}$$

with

$$\begin{aligned}C_1 := &\Big\{\left[2-2\cosh(2h/l)-\cosh(h/l)+\cosh(3h/l)\right]\left[Khl-2Khl^3\eta_1Z_1+\left(\omega_u+\omega_l\right)\eta_2l\right] + \\ &+\left[\sinh(3h/l)-3\sinh(h/l)\right]\left(V_0\eta_1+Kh^2l^2\eta_1Z_1\right)\Big\}\Big/\Big\{2hl^2\eta_1Z_1\left[\sinh(3h/l)-3\sinh(h/l)\right]+ \\ &+\left[2-2\cosh(2h/l)-\cosh(h/l)+\cosh(3h/l)\right]\left(2l-4l^3\eta_1Z_1\right)\Big\}.\end{aligned} \tag{5.24}$$

Equation (5.22) together with Eqs (5.23) and (5.24) is the analytical solution to our boundary value problem. The microrotation $\omega_3$ can now easily be calculated by means of Eqs (5.16) and (5.22) and is given as

$$\begin{aligned}\omega_3 = &\frac{C_1-K\,x_2}{\eta_2}+ \\ &-\frac{\eta_1}{\eta_2}\left\{2Z_1l^2\left(C_1-K\,x_2\right)+l\left[Z_1K\left(2l^2+h^2\right)-2Z_1C_1h+C_2\right]\frac{\cosh(x_2/l)}{\sinh(h/l)}+\right. \\ &\left.+l\left(C_2+2l^2Z_1K\right)\left[\sinh(x_2/l)-\cosh(x_2/l)\frac{\cosh(h/l)}{\sinh(h/l)}\right]+\frac{V_0}{l}\frac{\cosh(x_2/l)}{\sinh(h/l)}\right\}.\end{aligned} \tag{5.25}$$

Before we proceed, let us briefly discuss the identification of the material parameters in the constitutive model. The best way would surely be to measure directly the velocity fields of the ERF for different electric fields and field directions (for example: $E_1^2-E_2^2=0$, $E_1\neq 0$, $E_2=0$, $E_2\neq 0$, $E_1=0$ and so on). In reality, this seems to be unapplicable. However, an acceptable compromise would be to measure the flow rate vs. the pressure drop for different electric fields and field directions. In a simpler case without electric fields, this method has already been pointed out by Stokes in (1984) (see for example p.47).

### 5.1. Channel and shear flows

Let us now illustrate the above solutions for a specific set of parameter values. It turns out that the „perfect" parameter choice is not easy to find. As is well-known, a great variety of velocity profiles is possible. The following pictures have been generated by the following set of parameters

$$\bar{\alpha}_2 = 0.3, \quad \bar{\alpha}_3 + \bar{\alpha}_4 = 1/4, \quad \bar{\alpha}_5 = 1/16, \quad \bar{\alpha}_7 - \bar{\alpha}_9 = 1/4, \quad \bar{\alpha}_8 = -7/16,$$

$$\bar{\beta}_8 + \bar{\beta}_{12} = 0, \quad \bar{\beta}_9 + \bar{\beta}_{13} = 0.1, \quad \bar{\beta}_{10} + \bar{\beta}_{11} - \bar{\beta}_{14} - \bar{\beta}_{15} = 0.01.$$

This choice was partly guided by the restrictions imposed upon the parameters by the entropy inequality, compare especially Eq.$(4.6)_{2,\,3}$. The signs of $\bar{\alpha}_8$ and $\bar{\alpha}_5$ were chosen due to the following reason. Comparing Eqs (5.13) and (5.14) it can be seen that $\bar{\alpha}_8 < 0$ and $\bar{\alpha}_5 > 0$ helps to ensure that the shear stress $S_{12}^{E_2}$ is larger than $S_{12}^{E_1}$ for the same electric field strength $E_0$; this reflects our impression that the shear stress should be larger if the electric field is perpendicular to the flow direction than if it is parallel to it[12].

Furthermore, we have chosen $\omega_u = \omega_l = 0$. Although investigations show that these values may have a strong influence on the velocity profiles, we want to point out that for the desired purpose here we do not necessarily need dependences of the type $\omega_u = f_u([\boldsymbol{E}]^2)$ and $\omega_l = f_l([\boldsymbol{E}]^2)$, respectively.

Let us now introduce two dimensionless quantities a $a \in [0, 1]$ and $b > 0$ which are defined via

$$\frac{E_1}{E_0} = ab, \quad \frac{E_2}{E_0} = b\sqrt{1-a^2}, \quad \frac{E_1^2 + E_2^2}{E_0^2} = b^2, \quad \frac{E_1^2 - E_2^2}{E_0^2} = (2a^2 - 1)b^2.$$

$b$ is the relative electric field strength, while $a$ is a „direction parameter": for the viscometric flows investigated here, $a = 0$ and $a = 1$, respectively, reflect the situations when the electric field is perpendicular and parallel to the flow direction, respectively.

In Fig.1a, the velocity profile is illustrated as a function of the channel coordinate $x_2$ and the direction parameter $a$. One clearly sees that the maximum velocity increases if $a$ increases, i.e., if the angle between the electric field and the flow direction decreases from *90* to *0* degrees. This reflects the desired fact that the flow is hindered much less if the electric field is parallel to the flow than if it is perpendicular to it. In Fig.1b, the maximum velocity is illustrated as a function of both $a$ and $b$. Firstly, the maximum velocity decreases if the electric field strength $b$ increases, as it is common for electrorheological fluids. Secondly, it shows that with increasing $b$ the dependence on $a$ is more pronounced. This means that for small electric field strengths the dependence of the flow on the direction of the electric field is less pronounced than for strong electric field strengths.

[12] This partly assumes that the sign of $\nu_{1,2}$ is the same as that of $(1/2)\nu_{1,2} + \omega_3$ which cannot be guaranteed for all cases. However, the results that have been found using the introduced set of parameters will show that this choice is reasonable.

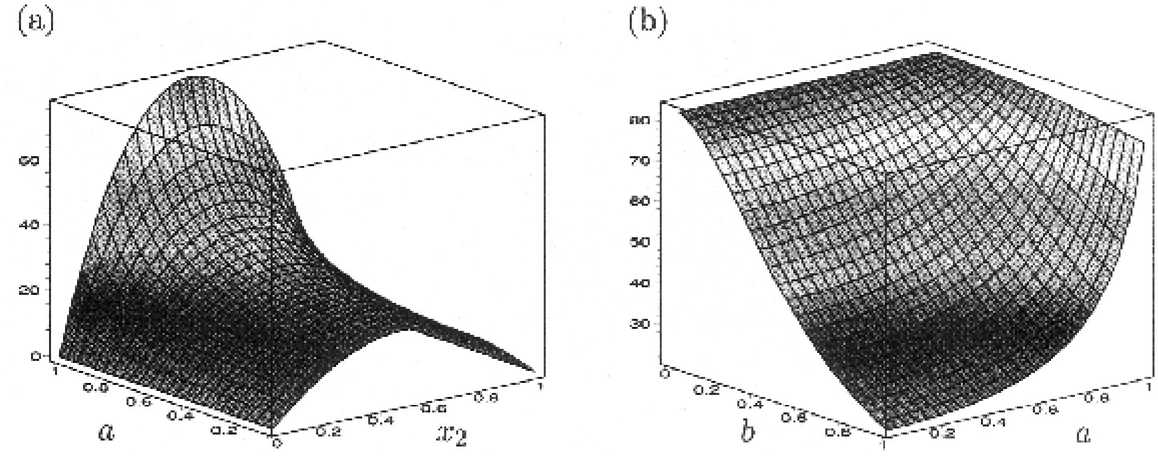


Fig.1. (a): Velocity profile $v_1$ depending on the direction parameter $a$. (b): Maximum velocity $v_1^{max}$ as a function of the relative electric field strength $b$ and the direction parameter $a$.

Finally, let us switch to the illustrations of the flow rate $Q$ which is given as

$$Q := \int_0^h v_1(x_2)\, d\, x_2 \,.$$

In Fig.2a, the flow rate is shown as a function of the pressure gradient $K$ and the direction parameter $a$. Due to the linear constitutive relation, the flow rate is a linear function of $K$. It increases with increasing $a$, as was to be expected from the velocity plots in Fig.1. Furthermore it should be noted that the dependence on $a$ is much more pronounced for higher than smaller pressure gradients $K$. In Fig.2b, the flow rate $Q$ is diplayed vs. the relative electric field strength $b$ and the direction parameter $a$. It can be seen that $Q$ increases either with decreasing $b$ or increasing $a$, as was to be expected from the velocity plots.

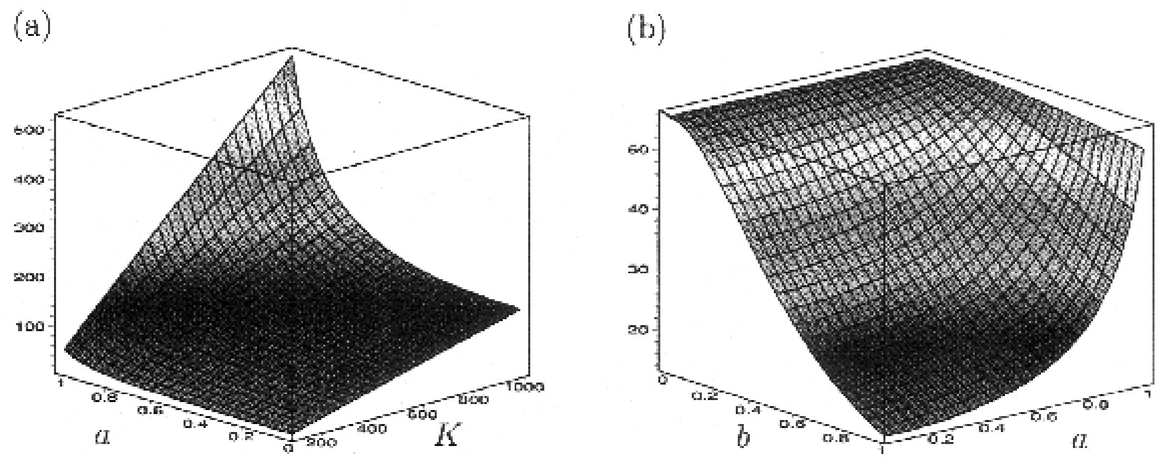


Fig.2. (a): Flow rate $Q$ as a function of the pressure gradient $K$ and the direction parameter $a$. (b): Flow rate $Q$ as a function of the relative electric field strength $b$ and the direction parameter $a$.

Note that the simple dependencies shown in Figs 1-2 are a direct consequence of the simple constitutive equation (linear in the shear rate and quadratic in the electric field). It is to be expected that more complicated (possibly non-monotonous) dependencies on the shear rate and/or the electric field will offer a great variety of possibilities to describe the behaviour of an electrorheological fluid more realistically.

## 6. Summary and concluding remarks

In this paper, we derived a micropolar theory for electrorheological fluids, starting with the thermomechanical and electromagnetic balance equations including the second law of thermodynamics in the form of the Clausius-Duhem inequality. Furthermore, we simplified the balance equations in view of electrorheological applications using an appropriate non-dimensionalization with a subsequent approximation. We then introduced constitutive equations both for the Cauchy and the couple stress tensor and evaluated the restrictions imposed on the material parameters by the entropy inequality. Linear constitutive equations were proposed which were discussed in a study of a viscometric flow. Finally, we illustrated the velocity and the flow rate depending on the electric field strength and the direction of the electric field.

The tasks of this paper were as follows. Firstly, we wanted to establish a complete framework for micropolar electrorheological fluids including all necessary balance equations, approximations and general constitutive equations (Sects. 1 – 4). In particular, the constitutive equations and the evaluation of the restrictions imposed on the material parameters by the second law of thermodynamics using scaling arguments (Sect. 4) may serve as a foundation for further studies on micropolar electrorheological constitutive equations.

Secondly, the main task of this paper was to show that the micropolar theory offers the possibility of describing the dependence of the electrorheological effect on the direction of the electric field in an objective and precise manner based on a sound theory, namely the framework of rational thermodynamics. In Section 5 we explicitly showed that the velocity (and hence the flow rate) depend on the direction and the absolute value of the electric field which enlarges the possibilities of describing electrorheological fluids in real applications significantly. To our knowledge, this has not been done before. Changing the dependence of the stress tensors on the shear rate and the electric field may easily be possible and results in a great varity of descriptive possibilities of electrorheological fluids. Thus, this paper may just be the beginning of a discussion on micropolar electrorheological constitutive equations.

## Acknowledgment

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## Nomenclature

$a$ – direction parameter of the electric field, $a \in [0, 1]$
$b$ – relative electric field strength, $b > 0$
$\boldsymbol{B}$ – magnetic flux density
$c$ – speed of electromagnetic waves in vacuo
$c_v$ – specific heat
$C_i$ – integration constants
$\boldsymbol{D}$ – symmetric part of the velocity gradient
$\boldsymbol{D}^e$ – dielectric displacement
$e$ – specific internal energy
$e_i, i = 1, 2, 3$ – fixed Cartesian basis
$\boldsymbol{E}$ – electric field
$\boldsymbol{f}$ – mechanical force density

$\boldsymbol{f}^{e}$ – electromagnetic force density
$f^{E}$, $f^{D}$ – equilibrium and non-equilibrium parts of $f$
$f_0$ – appropriate characteristic quantity of $f$
$\boldsymbol{G}$ – auxiliary quantity defined by $(\nabla \mathrm{w})_S = \frac{1}{3}(tr(\nabla \mathrm{w})_S)\boldsymbol{I} + \boldsymbol{G}$, $tr\,\boldsymbol{G} = 0$
$h$ – channel height
$\boldsymbol{H}$ – magnetic field
$\boldsymbol{I}$ – identity matrix
$\boldsymbol{J}$ – current density
Á – conductive current density
$k$ – thermal conductivity
$K$ – (constant) pressure gradient
$\boldsymbol{l}$ – mechanical couple density
$\boldsymbol{l}^{e}$ – electromagnetic couple density
$l$ – abbreviation (internal lengthscale)
$\boldsymbol{L}$ – velocity gradient
$\boldsymbol{M}$ – magnetization
$M$ – effective magnetization
$\boldsymbol{N}$ – couple stress tensor
$\boldsymbol{P}$ – electric polarization
$\boldsymbol{q}$ – heat flux
$q^{e}$ – density of the free electric charges
$Q$ – flow rate
$\boldsymbol{Q}$ – time independent orthogonal tensor (Galilean transformation)
Re – Reynolds number
$s$ – specific internal spin
$\boldsymbol{S}$ – extra stress tensor
Str – Strouhal number
$\boldsymbol{T}$ – Cauchy stress tensor
$\boldsymbol{v}$ – material velocity
$\boldsymbol{v}_0$, $\boldsymbol{b}_0$ – constant vectors (Galilean transformation)
$V_0$ – constant velocity
$w$ – mechanical energy supply density
$w^{e}$ – electromagnetic energy production density
$\boldsymbol{W}$ – skewsymmetric part of the velocity gradient
$\boldsymbol{x}$, $\boldsymbol{X}$, $t$ – coordinates, time
$Z_1$ – abbreviation
$\alpha$ – constant, $\alpha \in [0, 1]$
$\alpha_i$, $\overline{\alpha}_i$ – material parameters of $\boldsymbol{S}$
$\beta_i$, $\overline{\beta}_i$ – material parameters of $\boldsymbol{N}^{D}$
$\gamma$ – auxiliary quantity used for scaling
$\varepsilon$ – small non-dimensional number
e – isotropic third order tensor
$\varepsilon_0$, $\mu_0$ – dielectric constant and permeability in vacuo
E – effective electric field strength
$\zeta_i$ – abbreviations (containing the material parameters $\overline{\beta}_i$)
$\eta$ – specific entropy
$\eta_i$ – abbreviations (containing the material parameters $\overline{\alpha}_i$)
$\theta$ – absolute temperature
Q – symmetric micro-inertia tensor

$\mathrm{X}$ – proper orthonormal tensor, rotation
$\pi$ – thermodynamic pressure
$\rho$ – mass density
$\psi$ – specific free energy
$\mathrm{w}$ – microrotational velocity vector
$\mathrm{W}$ – microrotational velocity tensor
$\nabla$ – derivative with respect to $\boldsymbol{x}$
$(\nabla \mathrm{w})_S, (\nabla \mathrm{w})_A$ – symmetric and skewsymmetric part of $\nabla \mathrm{w}$

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