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FREE TRANSVERSE VIBRATIONS OF MONOCLINIC RECTANGULAR PLATES WITH CONTINUOUSLY VARYING THICKNESS AND DENSITY

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Free transverse vibrations of a rectangular plate composed of a monoclinic elastic material are discussed. The plate is considered to be exponentially varying in density and thickness along one direction of the plate. Hamilton's principle is used to derive the equation of motion and its solution is obtained using Chebyshev collocation technique. Frequency equations are derived for three boundary value problems when two edges of the plate are simply supported and the other two have different possibilities, namely (i) $C – S – C - S$ (ii) $C – S - S - S$ (iii) $C – S – F – S$, where C, S and F denote the clamped, simply supported and free edge respectively. Effects of thickness and density variation on modes of vibrations have been analyzed. Numerical computations have been performed for a specific model of monoclinic plates and the results obtained are compared with those for orthotropic plates already given in Lal (2003).

Key words: transverse vibrations, monoclinic, Chebyshev collocation method and frequency equation.

1. Introduction

Modern engineering structures are based on different types of design, which involve various types of anisotropic and non-homogeneous materials in the form of their structure components. Depending upon the requirement, durability and reliability, materials are being developed so that they can be used to give better strength and efficiency. The equipment used in air-jet, communications and in other similar technological industries take into consideration such materials, which not only reduce the weight and size but also are reliable in terms of efficiency, strength and economy. Various problems of free vibrations of plate made up of an elastic material with different boundary conditions have been discussed e.g., see Gorman (1982), Sizlard (1974) and Leissa (1969; 1973) among others. They have used different methods to find out the frequency equation of the modes of propagation. Appl and Byers (1965) determined the fundamental frequencies for simply supported rectangular plates of linearly varying thickness. Jain and Soni (1973) attempted a problem of free transverse vibrations of rectangular plates of parabolically varying thickness on the basis of classical theory of plates. They obtained the solution of the equation of motion using Frobenious method and derived the frequency equation for the plate whose two edges are simply supported. Biswas (1978) discussed large deflection of a heated orthotropic rectangular plate. He derived the governing equations using Berger's assumption and determined the deflection for a simply supported plate. De (1981) discussed the problem of vibrations of monoclinic crystal plates. Ng and Araar (1989) studied free vibration and buckling analysis of clamped rectangular plates of variable thickness using Galerkin method. Sonzogni *et al*. (1990) discussed free vibrations of rectangular plates of exponentially varying thickness using the

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optimized Kantorovich method and Finite Element Method. Bhat *et al*. (1990) discussed numerical experiments on the determination of natural frequencies of transverse vibrations of rectangular plates of non-uniform thickness. Bhat (1991) studied vibrations of rectangular plates on point and line support using characteristic orthogonal polynomials of the Rayleigh-Ritz method. Grossi and Bhat (1995) studied natural frequencies of edge restrained tapered rectangular plates. Sakata *et al*. (1996) discussed natural frequencies of orthotropic rectangular plates obtained by on iterative reduction of the partial differential equation. Rajalingham *et al*. (1996) studied vibrations of rectangular plates using plate characteristic functions as shape functions according to Rayleigh-Ritz method. Lal *et al*. (1996) studied free transverse vibrations of a thin rectangular plate of exponentially varying thickness and resting on an elastic foundation of Winkler type. Using Levy's technique, they derived characteristic equations for three different combinations of the boundary conditions at the other two edges. Rajalingham *et al*. (1997) attempted a problem of vibrations of rectangular plates by reducing the partial differential equation into simultaneous ordinary equations. Bespalova and Kitaigorodskii (2001) studied features of the free planer vibrations of orthotropic rectangular plates. They discussed the spectrum of the symmetric planer vibrations of an orthotropic rectangular plate as a function of the characteristics of orthotropic materials and its dimensions. Hui and Haun-ran (2001) discussed natural frequency of rectangular orthotropic corrugated –core sandwich plates with all edges simply supported. They presented a simple approach to reduce the governing equations for orthotropic corrugated core sandwich plates to a single equation containing only one displacement function and obtained the exact solutions of the natural frequencies. Taylor and Govindji (2002) discussed the solution of clamped rectangular plate problems. The method used by them is based upon the classical double cosine series expansion and an implementation of the Sherman–Morrison-Woodbury formula. Numerical solutions of rectangular plates with various side ratios are presented and compared with the solution generated via Hencky's method. Recently, Lal (2003) discussed a problem of transverse vibrations of orthotropic non-uniform rectangular plates with continuously varying density on the basis of the classical plate theory. He obtained the equation of motion and solved it using Chebyshev polynomials and derived frequency equations for three different combinations of clamped, simply supported and free boundary conditions at the other two edges of the plate.

2. Problem and derivation of equation of motion

In rectangular Cartesian co-ordinates (x, y, z), we consider a rectangular plate of a monoclinic material of length 'a', breadth 'b' and thickness $h = h(x, y)$ such that the middle surface of the plate is along $z = 0$ and the origin is at one of the corners of the plate. The z-axis is taken perpendicular to the plate. Let the components of displacement (u, v, w) be along the Cartesian axes. Following Sizlard (1974), the relation between these components of displacement are given by

$$u = -z\frac{\partial w}{\partial x}, \qquad v = -z\frac{\partial w}{\partial y}. \tag{2.1}$$

Also the strain components are given by

$$\begin{aligned} &e_{xx} = \frac{\partial u}{\partial x}, \qquad e_{yy} = \frac{\partial v}{\partial y}, \qquad e_{zz} = \frac{\partial w}{\partial z}, \\ &e_{yz} = \frac{\partial w}{\partial y} + \frac{\partial v}{\partial z}, \qquad e_{zx} = \frac{\partial u}{\partial z} + \frac{\partial w}{\partial x}, \qquad e_{xy} = \frac{\partial v}{\partial x} + \frac{\partial u}{\partial y}. \end{aligned} \tag{2.2}$$

Stress-strain relations for a monoclinic material are given by

$$t_{xx} = c_{11}e_{xx} + c_{12}e_{yy} + c_{13}e_{zz} + c_{14}e_{yz}, \qquad t_{yy} = c_{21}e_{xx} + c_{22}e_{yy} + c_{23}e_{zz} + c_{24}e_{yz},$$

$$t_{zz} = c_{31}e_{xx} + c_{32}e_{yy} + c_{33}e_{zz} + c_{34}e_{yz}, \qquad t_{yz} = c_{41}e_{xx} + c_{42}e_{yy} + c_{43}e_{zz} + c_{44}e_{yz}, \tag{2.3}$$

$$t_{xz} = c_{55}e_{zx} + c_{56}e_{xy}, \qquad t_{xy} = c_{65}e_{zx} + c_{66}e_{xy}$$

where $c_{ij}(i, j = 1,2,3,...,6)$ are elastic constants and other symbols have their usual meanings. The strain energy V is given by

$$dV = \frac{1}{2}\left(t_{xx}e_{xx} + t_{yy}e_{yy} + t_{zz}e_{zz} + t_{yz}e_{yz} + t_{zx}e_{zx} + t_{xy}e_{xy}\right). \tag{2.4}$$

Putting Eqs (2.1) to (2.3) into Eq.(2.4) and integrating over the limits of the plate considered, we obtain

$$V = \frac{h^3}{24}\int_0^a\int_0^b\left[c_{11}\left(\frac{\partial^2 w}{\partial x^2}\right)^2 + (c_{12}+c_{21})\frac{\partial^2 w}{\partial x^2}\frac{\partial^2 w}{\partial y^2} + c_{22}\left(\frac{\partial^2 w}{\partial y^2}\right)^2 + 4c_{66}\left(\frac{\partial^2 w}{\partial x \partial y}\right)^2\right]dx\,dy. \tag{2.5}$$

The kinetic energy T of the plate is given by

$$T = \frac{1}{2}h\rho\int_0^a\int_0^b\left(\frac{\partial w}{\partial t}\right)^2 dx\,dy. \tag{2.6}$$

The variation of V is given by

$$\begin{aligned}\delta V = \frac{h^3}{12}\int_0^a\int_0^b\Bigg[&c_{11}\frac{\partial^2 w}{\partial x^2}\frac{\partial^2 \delta w}{\partial x^2} + (c_{12}+c_{21})\left(\frac{\partial^2 w}{\partial x^2}\frac{\partial^2 \delta w}{\partial y^2} + \frac{\partial^2 w}{\partial y^2}\frac{\partial^2 \delta w}{\partial x^2}\right) + \\ &+ c_{22}\frac{\partial^2 w}{\partial y^2}\frac{\partial^2 \delta w}{\partial y^2} + 4c_{66}\frac{\partial^2 w}{\partial x \partial y}\frac{\partial^2 \delta w}{\partial x \partial y}\Bigg]dx\,dy,\end{aligned} \tag{2.7}$$

and the variation of T is given by

$$\delta T = h\rho\int_0^a\int_0^b\frac{\partial\,\delta w}{\partial t}\frac{\partial w}{\partial t}dx\,dy. \tag{2.8}$$

We shall now obtain the equation of motion using Hamilton's principle given by

$$\delta\int_{t_1}^{t_2} L\,dt = 0 \tag{2.9}$$

where t_1 and t_2 are initial and final values of time and $L = T - V$ is the Lagrangian. Substituting Eqs (2.7) and (2.8) into Eq.(2.9), we get

$$\int_0^a\int_0^b\int_{t_1}^{t_2}\left[\rho h\frac{\partial w}{\partial t}\frac{\partial \delta w}{\partial t} - \frac{h^3}{12}\left\{c_{11}\frac{\partial^2 w}{\partial x^2}\frac{\partial^2 \delta w}{\partial x^2} - c_{22}\frac{\partial^2 w}{\partial y^2}\frac{\partial^2 \delta w}{\partial y^2} - (c_{12}+c_{21})\times\right.\right.$$
$$\left.\left.\times\left(\frac{\partial^2 w}{\partial x^2}\frac{\partial^2 \delta w}{\partial y^2} + \frac{\partial^2 w}{\partial y^2}\frac{\partial^2 \delta w}{\partial x^2}\right) - 4c_{66}\frac{\partial^2 w}{\partial x\partial y}\frac{\partial^2 \delta w}{\partial x\partial y}\right\}\right]dx\,dy\,dt = 0 \tag{2.10}$$

Performing the integration in Eq.(2.10) with respect to 't' and equating the coefficient of δw equal to zero, we get

$$c_{11}\frac{h^3}{12}\frac{\partial^4 w}{\partial x^4} + c_{22}\frac{h^3}{12}\frac{\partial^4 w}{\partial y^4} + 2(c_{12}+c_{21}+c_{66})\frac{h^3}{12}\frac{\partial^4 w}{\partial x^2\partial y^2} +$$
$$+2\frac{\partial}{\partial x}\left\{(c_{12}+c_{21}+c_{66})\frac{h^3}{12}\right\}\frac{\partial^3 w}{\partial x\partial y^2} + 2\frac{\partial}{\partial y}\left\{(c_{12}+c_{21}+c_{66})\frac{h^3}{12}\right\}\frac{\partial^3 w}{\partial y\partial x^2} +$$
$$+2\frac{\partial}{\partial y}\left(c_{22}\frac{h^3}{12}\right)\frac{\partial^3 w}{\partial y^3} + \frac{\partial^2}{\partial x^2}\left(c_{11}\frac{h^3}{12}\right)\frac{\partial^2 w}{\partial x^2} + \frac{\partial^2}{\partial y^2}\left(c_{22}\frac{h^3}{12}\right)\frac{\partial^2 w}{\partial y^2} + \tag{2.11}$$
$$\frac{\partial^2}{\partial y^2}\left\{(c_{12}+c_{21})\frac{h^3}{12}\right\}\frac{\partial^2 w}{\partial x^2} + \frac{\partial^2}{\partial x^2}\left\{(c_{12}+c_{21})\frac{h^3}{12}\right\}\frac{\partial^2 w}{\partial y^2} +$$
$$+4\frac{\partial^2}{\partial y\partial x}\left(c_{66}\frac{h^3}{12}\right)\frac{\partial^2 w}{\partial y\partial x} + \rho h\frac{\partial^2 w}{\partial t^2} = 0.$$

This is the required equation of motion.

Let us assume that two opposite edges of the plate given by $y = 0$ and $y = b$ are simply supported and that the thickness is independent of y i.e., $h = h(x)$. For harmonic vibrations, the deflection function w (Levy approach) is assumed to be

$$w(x,y,t) = \overline{w}(x)\sin(p\pi y/b)\exp(\iota\omega t) \tag{2.12}$$

where 'p' is a positive integer and ω is the radian frequency.

Substituting (2.12) into Eq.(11), we get

$$h^3\frac{d^4\overline{w}}{dx^4} + 6h^2h'\frac{d^3\overline{w}}{dx^3} + \left\{3h^2h'' + 6hh'^2 - 2\frac{(c_{12}+c_{21}+2c_{66})}{c_{11}}\frac{p^2\pi^2}{b^2}h^3\right\}\frac{d^2\overline{w}}{dx^2} +$$
$$-6\frac{(c_{12}+c_{21}+2c_{66})}{c_{11}}\frac{p^2\pi^2}{b^2}h^2h'\frac{d\overline{w}}{dx} + \left\{\frac{c_{22}}{c_{11}}\frac{p^4\pi^4}{b^4}h^3 - 3\frac{(c_{12}+c_{21})}{c_{11}}\frac{p^2\pi^2}{b^2}\times\right. \tag{2.13}$$
$$\left.\times\left(h^2h'' + 2hh'^2\right) - \frac{12h\omega^2}{c_{11}}\right\}\overline{w} = 0.$$

Introducing the following non-dimensional variables

$$X = x/a\,, \qquad Y = y/b\,, \qquad \bar{h} = h/a\,, \qquad W = \bar{w}/a\,.$$

Equation (2.13) reduces to

$$\begin{gathered}\bar{h}^3 \frac{d^4W}{dX^4} + 6\bar{h}^2\bar{h}' \frac{d^3W}{dX^3} + \left\{ 3\bar{h}^2\bar{h}'' + 6\bar{h}\,\bar{h}'^2 - 2\frac{(c_{12}+c_{21}+2c_{66})}{c_{11}}\frac{p^2\pi^2a^2}{b^2}\bar{h}^3 \right\}\frac{d^2W}{dX^2} + \\ -6\frac{(c_{12}+c_{21}+2c_{66})}{c_{11}}\frac{p^2\pi^2a^2}{b^2}\bar{h}^2\bar{h}'\frac{dW}{dX} + \left\{ \frac{c_{22}}{c_{11}}\frac{p^4\pi^4a^4}{b^4}\bar{h}^3 - 3\frac{(c_{12}+c_{21})}{c_{11}}\frac{p^2\pi^2a^2}{b^2} \times \right. \\ \left. \times\left(\bar{h}^2\bar{h}'' + 2\bar{h}\,\bar{h}'^2\right) - \frac{12\rho\bar{h}a^2\omega^2}{c_{11}} \right\} W = 0\end{gathered} \tag{2.14}$$

where prime denotes the differentiation with respect to X.

We assume that the thickness and density of the plate are varying exponentially along the X-direction and are given by $\bar{h} = h_0 e^{\alpha X}$ and $\rho = \rho_0 e^{\beta X}$, where h_0 and ρ_0 are the thickness and density respectively of the plate at the end $X = 0$, α and β are the taper and density parameters respectively.

With these variations, Eq.(2.14) reduces to

$$B_0\frac{d^4W}{dX^4} + B_1\frac{d^3W}{dX^3} + B_2\frac{d^2W}{dX^2} + B_3\frac{dW}{dX} + B_4 W = 0 \tag{2.15}$$

where

$$B_0 = 1\,, \qquad B_1 = 6\alpha\,,$$

$$B_2 = 9\alpha^2 - 2\frac{(c_{12}+c_{21}+2c_{66})}{c_{11}}\frac{p^2\pi^2a^2}{b^2}\,, \qquad B_3 = -6\frac{(c_{12}+c_{21}+2c_{66})}{c_{11}}\frac{p^2\pi^2a^2}{b^2}\alpha\,,$$

$$B_4 = \frac{c_{22}}{c_{11}}\frac{p^4\pi^4a^4}{b^4} - 9\frac{(c_{12}+c_{21})}{c_{11}}\frac{p^2\pi^2a^2}{b^2}\alpha^2 - \Omega^2 e^{(\beta-2\alpha)X}\,, \qquad \Omega^2 = \frac{12\rho_0 a^2\omega^2}{c_{11}h_0^2}$$

3. Application of the method

We find the solution of Eq.(2.15) using the Chebyshev collocation technique. Taking a new independent variable ϕ defined by $\phi = 2X - 1$, we see that the range $0 \le X \le 1$ is transformed into $-1 \le \phi \le 1$. With this transformation, Eq.(2.15) reduces to

$$V_0\frac{d^4W}{d\phi^4} + V_1\frac{d^3W}{d\phi^3} + V_2\frac{d^2W}{d\phi^2} + V_3\frac{dW}{d\phi} + V_4 W = 0 \tag{3.1}$$

where $V_i = 2^{4-i} B_i, \quad (i = 0, 1, 2, 3, 4)$.

In order to apply the Chebyshev collocation technique (see Fox, 1957; Fox and Parker, 1968; Snyder, 1969), we assume

$$W = d_1 + d_2 T_1 + d_3 T_1^{\ 1} + d_4 T_1^{\ 2} + \sum_{k=0}^{m-5} d_{k+5} T_k^{\ 4} \tag{3.2}$$

where $d_j (j = 1, 2, ..., m)$ are unknown constants, $T_k (k = 0, 1, 2, ..., m-5)$ are Chebyshev polynomials and T_k^j denotes the j^{th} integral of T_k with respect to ϕ.

Substituting the value of W given by Eq.(3.2) into Eq.(3.1) and putting

$$\phi_k = \cos\left(\frac{2k+1}{m-4}\frac{\pi}{2}\right), \qquad (k = 0, 1, 2, ..., m-5), \tag{3.3}$$

into the resulting equation, one can obtain a set of $(m-4)$ equations in $d_j (j = 1, 2, ..., m)$. This set of equations can be put in matrix form as

$$[B][D] = [0] \tag{3.4}$$

where $[B]$ is a matrix of order $(m-4) \times m$ and $[D]$ and $[0]$ are column vectors of order $m \times 1$ and $(m-4) \times 1$ respectively.

4. Boundary conditions and frequency equations

We shall consider the following four sets of boundary conditions:
(i) C-C conditions: Clamped at the edges $X = 0$ and $X = 1$.
(ii) C-S conditions: Clamped at the edge $X = 0$ and simply supported at the edge $X = 1$.
(iii) C-F conditions: Clamped at the edge $X = 0$ and free at the edge $X = 1$.

The boundary conditions at C, S and F edges are respectively

$$W = \frac{dW}{dX} = 0, \tag{4.1}$$

$$W = \frac{d^2W}{dX^2} - c'_{11}\frac{p^2\pi^2a^2}{b^2}W = 0, \tag{4.2}$$

and

$$\frac{d^2W}{dX^2} - c'_{11}\frac{p^2\pi^2a^2}{b^2}W = \frac{d^3W}{dX^3} - \left(\frac{c'_{11} + 4c_{66}}{c_{11}}\right)\frac{p^2\pi^2a^2}{b^2}\frac{dW}{dX} = 0 \tag{4.3}$$

where $c'_{11} = \dfrac{c_{12} + c_{21}}{c_{11}}$.

Applying the boundary conditions given by Eq.(4.1) for the *C-C* case to Eq.(3.2) after making use of the transformation $\phi = 2X - 1$ defined above, one obtains a set of four homogeneous equations in $d_j\left(j = 1, 2, ..., m\right)$. These equations together with the $\left(m-4\right)$ equations given by (3.4) constitute a system of m equations in m unknowns. These equations can be written as

$$\left[\frac{B}{B_{CC}}\right]\left[D\right] = \left[0\right] \tag{4.4}$$

where $\left[B_{CC}\right]$ is a matrix of order $4 \times m$ and $\left[0\right]$ is a column matrix of order $m \times 1$. $\left[B\right]$ is a matrix of order $\left(m-4\right) \times m$, whose elements are given by

$$b_{1,1} = V_4, \quad b_{1,2} = V_4 T_1 + V_3, \quad b_{1,3} = V_4 T_1^1 + V_3 T_1 + V_2,$$

$$b_{1,4} = V_4 T_1^1 + V_3 T_1^1 + V_2 T_1 + V_1, \quad b_{1,5} = V_4 T_1^3 + V_3 T_1^2 + V_2 T_1^1 + V_1 T_1 + V_0,$$

$$b_{1,6} = V_4 T_1^4 + V_3 T_1^3 + V_2 T_1^2 + V_1 T_1^1 + V_0 T_1, \quad b_{1,7} = V_4 T_2^4 + V_3 T_2^3 + V_2 T_2^2 + V_1 T_2^1 + V_0 T_2,$$

$$b_{1,8} = V_4 T_3^4 + V_3 T_3^3 + V_2 T_3^2 + V_1 T_3^1 + V_0 T_3, \quad b_{1,9} = V_4 T_4^4 + V_3 T_4^3 + V_2 T_4^2 + V_1 T_4^1 + V_0 T_4,$$

$$b_{1,10} = V_4 T_5^4 + V_3 T_5^3 + V_2 T_5^2 + V_1 T_5^1 + V_0 T_5, \quad b_{1,11} = V_4 T_6^4 + V_3 T_6^3 + V_2 T_6^2 + V_1 T_6^1 + V_0 T_6,$$

$$b_{1,12} = V_4 T_7^4 + V_3 T_7^3 + V_2 T_7^2 + V_1 T_7^1 + V_0 T_7, \quad b_{1,13} = V_4 T_8^4 + V_3 T_8^3 + V_2 T_8^2 + V_1 T_8^1 + V_0 T_8,$$

$$b_{1,14} = V_4 T_9^4 + V_3 T_9^3 + V_2 T_9^2 + V_1 T_9^1 + V_0 T_9, \quad b_{1,15} = V_4 T_{10}^4 + V_3 T_{10}^3 + V_2 T_{10}^2 + V_1 T_{10}^1 + V_0 T_{10}$$

where $V_i, \left(i = 0, 1, 2, 3, 4\right)$ are defined earlier.

Since $b_{1,j}, j = 1,2,...,m$ are given in the terms of ϕ, so putting the value of ϕ form Eq.(3.3), we get $b_{i,j}\left(i = 1,2,...,m-4\right)$.

$$B_{cc} = \begin{bmatrix} 1 & T_1 & T_1^{\;1} & T_1^{\;2} & T_0^{\;4} & T_1^{\;4} & T_2^{\;4} & T_3^{\;4} & T_4^{\;4} & T_5^{\;4} & T_6^{\;4} & T_7^{\;4} & T_8^{\;4} & T_9^{\;4} & T_{10}^{\;4} \\ 0 & 1 & T_1 & T_1^{\;1} & T_0^{\;3} & T_1^{\;3} & T_2^{\;3} & T_3^{\;3} & T_4^{\;3} & T_5^{\;3} & T_6^{\;3} & T_7^{\;3} & T_8^{\;3} & T_9^{\;3} & T_{10}^{\;3} \\ 1 & T_1 & T_1^{\;1} & T_1^{\;2} & T_0^{\;4} & T_1^{\;4} & T_2^{\;4} & T_3^{\;4} & T_4^{\;4} & T_5^{\;4} & T_6^{\;4} & T_7^{\;4} & T_8^{\;4} & T_9^{\;4} & T_{10}^{\;4} \\ 0 & 1 & T_1 & T_1^{\;1} & T_0^{\;3} & T_1^{\;3} & T_2^{\;3} & T_3^{\;3} & T_4^{\;3} & T_5^{\;3} & T_6^{\;3} & T_7^{\;3} & T_8^{\;3} & T_9^{\;3} & T_{10}^{\;3} \end{bmatrix}.$$

The first two rows of matrix B_{cc} are obtained from the boundary conditions (4.1), which must be satisfied at $\phi = -1$ and the last two rows are obtained from the same boundary conditions in (4.1) but these are satisfied at $\phi = 1$.

For a non-trivial solution of Eq.(4.4), we must have

$$\left|\frac{B}{B_{CC}}\right| = 0 . \tag{4.5}$$

This is the frequency equation for transverse modes of propagation in the said case. Similarly, from the boundary conditions (4.2) for *C-S* conditions, the frequency equation can be obtained as

$$\left|\frac{B}{B_{CS}}\right| = 0 \tag{4.6}$$

where

$$B_{CS} = \begin{bmatrix} 1 & T_1 & T_1^{\ 1} & T_1^{\ 2} & T_0^{\ 4} & T_1^{\ 4} & T_2^{\ 4} & T_3^{\ 4} & T_4^{\ 4} & T_5^{\ 4} & T_6^{\ 4} & T_7^{\ 4} & T_8^{\ 4} & T_9^{\ 4} & T_{10}^{\ 4} \\ 0 & 1 & T_1 & T_1^{\ 1} & T_0^{\ 3} & T_1^{\ 3} & T_2^{\ 3} & T_3^{\ 3} & T_4^{\ 3} & T_5^{\ 3} & T_6^{\ 3} & T_7^{\ 3} & T_8^{\ 3} & T_9^{\ 3} & T_{10}^{\ 3} \\ 1 & T_1 & T_1^{\ 1} & T_1^{\ 2} & T_0^{\ 4} & T_1^{\ 4} & T_2^{\ 4} & T_3^{\ 4} & T_4^{\ 4} & T_5^{\ 4} & T_6^{\ 4} & T_7^{\ 4} & T_8^{\ 4} & T_9^{\ 4} & T_{10}^{\ 4} \\ R_1 & R_2 & R_3 & R_4 & R_5 & R_6 & R_7 & R_8 & R_9 & R_{10} & R_{11} & R_{12} & R_{13} & R_{14} & R_{15} \end{bmatrix}$$

where

$$R_1 = -p_3 , \qquad R_2 = -p_3 T_1 , \qquad R_3 = 4 - p_3 T_1^1 ,$$

$$R_4 = 4T_1 - p_3 T_1^2 , \qquad R_5 = 4T_0^2 - p_3 T_0^4 , \qquad R_6 = 4T_1^2 - p_3 T_1^4 ,$$

$$R_7 = 4T_2^2 - p_3 T_2^4 , \qquad R_8 = 4T_3^2 - p_3 T_3^4 , \qquad R_9 = 4T_4^2 - p_3 T_4^4 ,$$

$$R_{10} = 4T_5^2 - p_3 T_5^4 , \qquad R_{11} = 4T_6^2 - p_3 T_6^4 , \qquad R_{12} = 4T_7^2 - p_3 T_7^4 ,$$

$$R_{13} = 4T_8^2 - p_3 T_8^4 , \qquad R_{14} = 4T_9^2 - p_3 T_9^4 , \qquad R_{15} = 4T_{10}^2 - p_3 T_{10}^4 ,$$

and $$p_3 = c'_{11} \frac{p^2 \pi^2 a^2}{b^2} .$$

The first two rows of matrix B_{CS} are obtained from the boundary conditions (4.1), which must be satisfied at $\phi = -1$ and the last two rows are obtained from the same boundary conditions given in (4.2) but these are satisfied at $\phi = 1$.

And from the boundary conditions (4.3) for *C-F* conditions, the frequency equation becomes

$$\left|\frac{B}{B_{CF}}\right| = 0 , \tag{4.7}$$

$$B_{CF} = \begin{bmatrix} 1 & T_1 & T_1^{\ 1} & T_1^{\ 2} & T_0^{\ 4} & T_1^{\ 4} & T_2^{\ 4} & T_3^{\ 4} & T_4^{\ 4} & T_5^{\ 4} & T_6^{\ 4} & T_7^{\ 4} & T_8^{\ 4} & T_9^{\ 4} & T_{10}^{\ 4} \\ 0 & 1 & T_1 & T_1^{\ 1} & T_0^{\ 3} & T_1^{\ 3} & T_2^{\ 3} & T_3^{\ 3} & T_4^{\ 3} & T_5^{\ 3} & T_6^{\ 3} & T_7^{\ 3} & T_8^{\ 3} & T_9^{\ 3} & T_{10}^{\ 3} \\ R_1 & R_2 & R_3 & R_4 & R_5 & R_6 & R_7 & R_8 & R_9 & R_{10} & R_{11} & R_{12} & R_{13} & R_{14} & R_{15} \\ S_1 & S_2 & S_3 & S_4 & S_5 & S_6 & S_7 & S_8 & S_9 & S_{10} & S_{11} & S_{12} & S_{13} & S_{14} & S_{15} \end{bmatrix}$$

where

$$S_1 = 0, \quad S_2 = p_4, \quad S_3 = p_4 T_1,$$

$$S_4 = 4 - p_4 T_1^1, \quad S_5 = 4T_0^1 - p_4 T_0^3, \quad S_6 = 4T_1^1 - p_4 T_1^3, \quad S_7 = 4T_2^1 - p_4 T_2^3,$$

$$S_8 = 4T_3^1 - p_4 T_3^3, \quad S_9 = 4T_4^1 - p_4 T_4^3, \quad S_{10} = 4T_5^1 - p_4 T_5^3, \quad S_{11} = 4T_6^1 - p_4 T_6^3,$$

$$S_{12} = 4T_7^1 - p_4 T_7^3, \quad S_{13} = 4T_8^1 - p_4 T_8^3, \quad S_{14} = 4T_9^1 - p_4 T_9^3, \quad S_{15} = 4T_{10}^1 - p_4 T_{10}^3,$$

and
$$p_4 = -\left(2 - c'_{11}\right)\frac{p^2\pi^2 a^2}{b^2}.$$

The first two rows of matrix B_{CF} are obtained from the boundary conditions (4.1), which must be satisfied at $\phi = -1$ and the last two rows are obtained from the same boundary conditions given in Eq.(4.3) but these are satisfied at $\phi = 1$.

5. Numerical results and discussion

Frequency Eqs (4.5)-(4.7) are solved numerically for various values of density parameter β, taper constant α and aspect ratio $d\left(=\frac{a}{b}\right)$ for first three modes of vibration only. In all computations we have taken $m = 15$, $p = 1$ and $h_0 = 0.1cm$. The values of elastic constants for rock gypsum as monoclinic material are taken as follows (Haussuhl, 1965)

$$c_{11} = 7.859 \times 10^6 \ erg/cm^3, \qquad c_{12} = c_{21} = 4.1 \times 10^6 \ erg/cm^3,$$

$$c_{22} = 6.287 \times 10^6 \ erg/cm^3, \qquad c_{66} = 1.044 \times 10^6 \ erg/cm^3.$$

Using these values, we have calculated the value of frequency parameter (Ω) from Eq.(4.5) (i.e., for *C-C* conditions of the plates) at different values of density parameter β, namely $\beta = -0.5$, *-0.3, -0.1, 0.0, 0.1, 0.3, 0.5, 1.0* and at two different values of taper parameter $\alpha = -0.5$ and $\alpha = 0.5$. The results obtained are depicted in Fig.1. In this figure, we have depicted only the curve of the first mode of vibrations in monoclinic and orthotropic plates for *C-C* conditions of the plates, however higher modes of propagation also exist. We note that at $\alpha = -0.5$ and $\beta = -0.5$, the value Ω is *26.37* for a monoclinic plate and *21.53* for an orthotropic plate, while at $\alpha = 0.5$ and $\beta = -0.5$, the value Ω is *42.79* for a monoclinic plate and *34.98* for an orthotropic plate. As the value of density parameter β increases from the value *–0.5* to *0.5*, the value of frequency parameter Ω decreases for the monoclinic as well as for orthotropic plates. Thus we conclude that the frequency of vibrations in the monoclinic plate is higher than that in the orthotropic plate, but have in the same pattern.

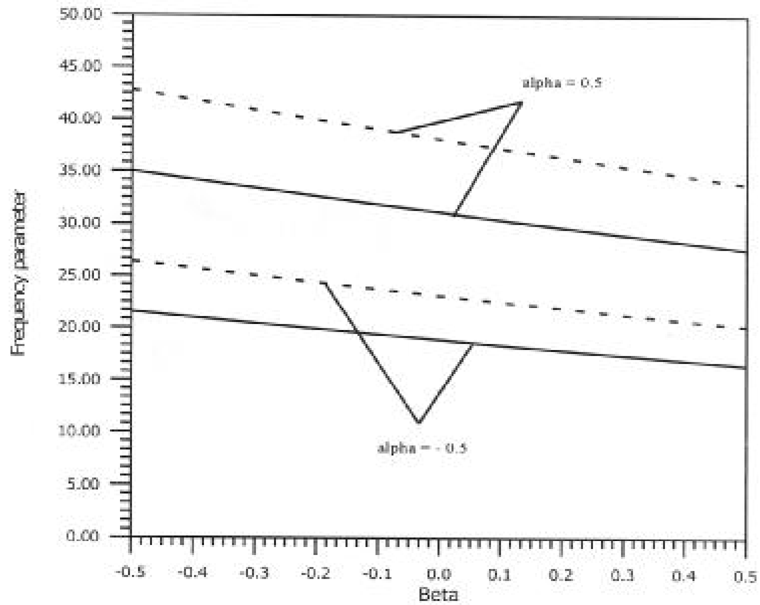

Fig1. (*C-C* conditions) Variation of first mode of frequency parameter with density parameter. (Soild curve – Orthotropic plate, Dashed curve – Monoclinic plate).

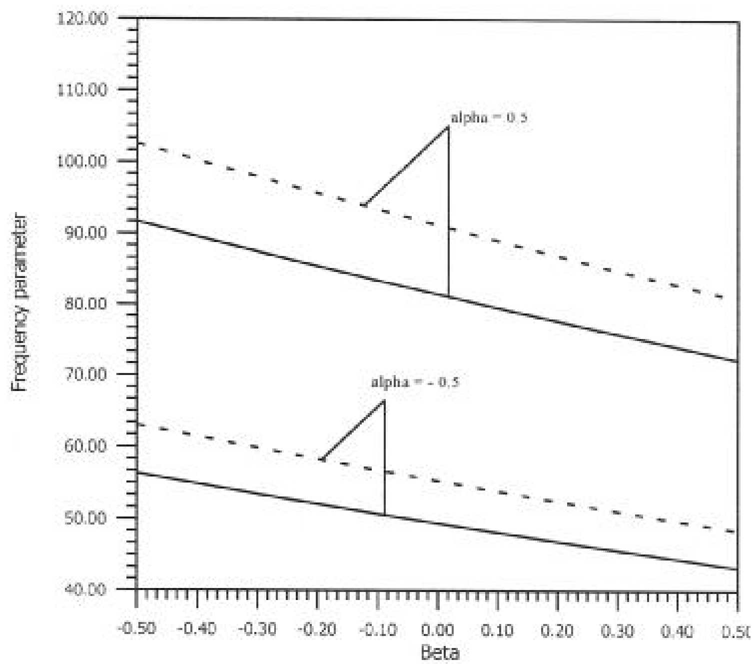

Fig.2. (*C-C* conditions) Variation of second mode of frequency parameter with density parameter. (Soild curve – Orthotropic plate, Dashed curve – Monoclinic plate).

In Figure 2, we note that at $\alpha = -0.5$ and $\beta = -0.5$, the value of Ω is *62.98* for a monoclinic plate and *56.26* for an orthotropic plate, while at $\alpha = 0.5$ and $\beta = -0.5$, the value of Ω is *102.55* for a monoclinic plate and *91.63* for an orthotropic plate (*C-C* conditions) for the second mode of vibration. As the value of density parameter β increases from the value *–0.5* to *0.5*, the value of frequency parameter Ω decreases for the monoclinic as well as for orthotropic plates. Thus we conclude that the frequency of vibrations in the monoclinic plate is higher than that in the orthotropic plate, but again in the same pattern as in the first mode.

In Figure 3, we note that at $\alpha = -0.5$ and $\beta = -0.5$, the value of Ω is *116.08* for a monoclinic plate and *108.53* for an orthotropic plate, while at $\alpha = 0.5$ and $\beta = -0.5$, the value of Ω is *189.23* for a monoclinic plate and *176.53* for an orthotropic plate (*C-C* conditions) for the third mode of vibration. Here also, the pattern of variation of frequency parameter is found to be similar as was in the first and second modes of vibrations.

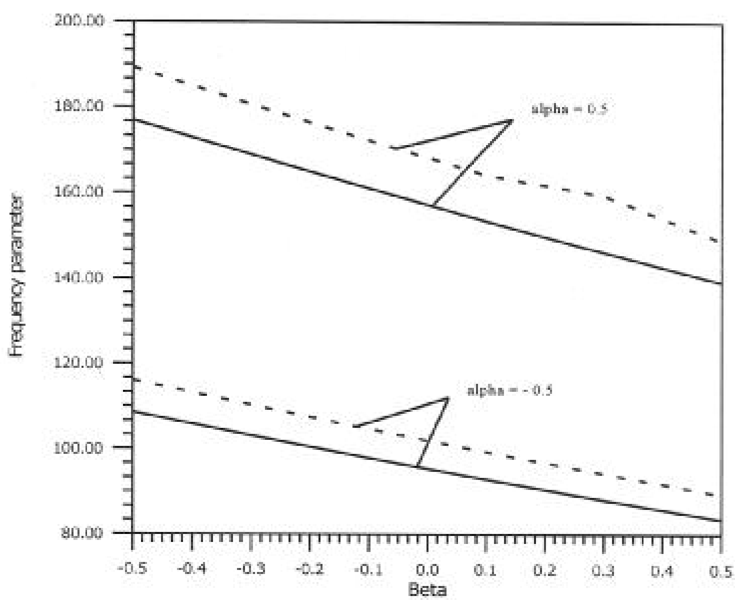

Fig.3. (*C-C* conditions) Variation of third mode of frequency parameter with density parameter. (Soild curve – Orthotropic plate, Dashed curve – Monoclinic plate).

Figure 4 depicts the variation of Ω with d when the plate is subjected to *C-C* conditions. At $d = 0.25$ with $\alpha = \beta = 0.5$, the values of Ω are *26.15, 70.93, 138.07* for a monoclinic plate and *25.75, 70.70, 137.45* for an orthotropic plate in the first, second, third modes of vibrations respectively. As the value of d increases through the values *0.25* to *2.0*, the values of frequency parameter Ω increase for the monoclinic as well as for orthotropic plates. Thus we conclude that the frequency of vibrations in the monoclinic plate is higher than that in the orthotropic plate.

Figures 5, 6 and 7 depict the same what is shown in Figs 1, 2 and 3 for *C-S* conditions of the plates. Here we conclude that the frequency of vibrations in the monoclinic plate is higher than that in the orthotropic plate. The results at $\beta = -0.5$ are summarized as follows:

Modes		Monoclinic	Orthotropic
I	$(\alpha = -0.5)$	*22.18*	*16.74*
	$(\alpha = 0.5)$	*35.43*	*25.05*
II	$(\alpha = -0.5)$	*54.39*	*47.11*
	$(\alpha = 0.5)$	*87.22*	*74.13*
III	$(\alpha = -0.5)$	*103.06*	*94.99*
	$(\alpha = 0.5)$	*166.18*	*152.08*

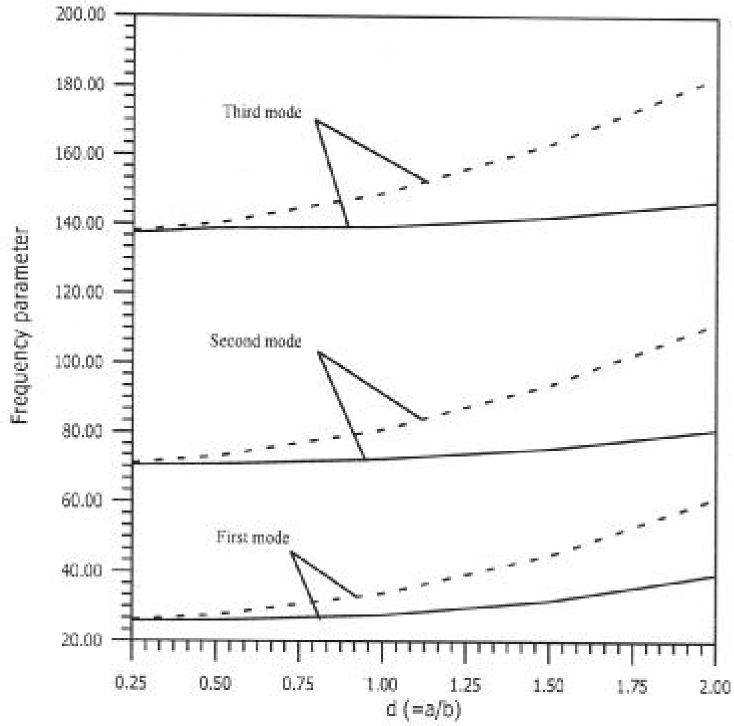

Fig.4. (*C-C* conditions) Variation of frequency parameters with aspect ratio. (Solid curve – orthotropic plate, Dashed curve – Monoclinic plate, $\alpha = \beta = 0.5$).

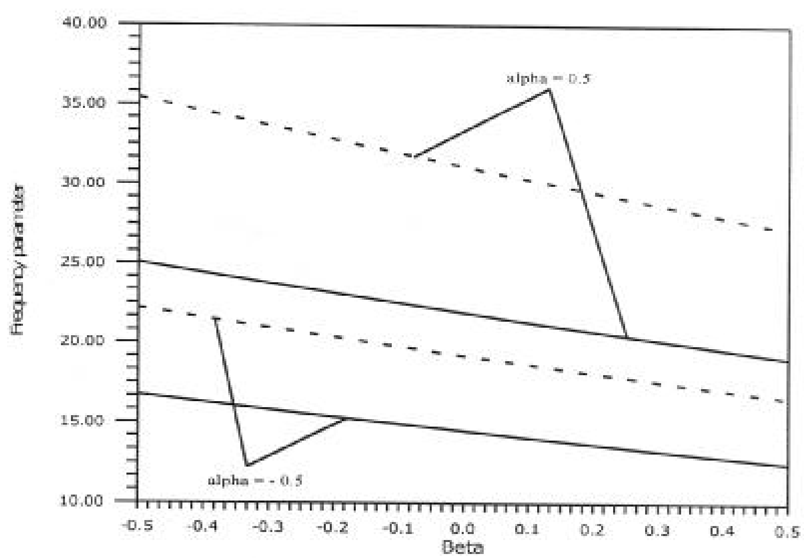

Fig.5. (*C-S* condition) Variation of first mode of frequency parameter with density parameter. (Soild curve – Orthotropic plate, Dashed curve – Monoclinic plate).

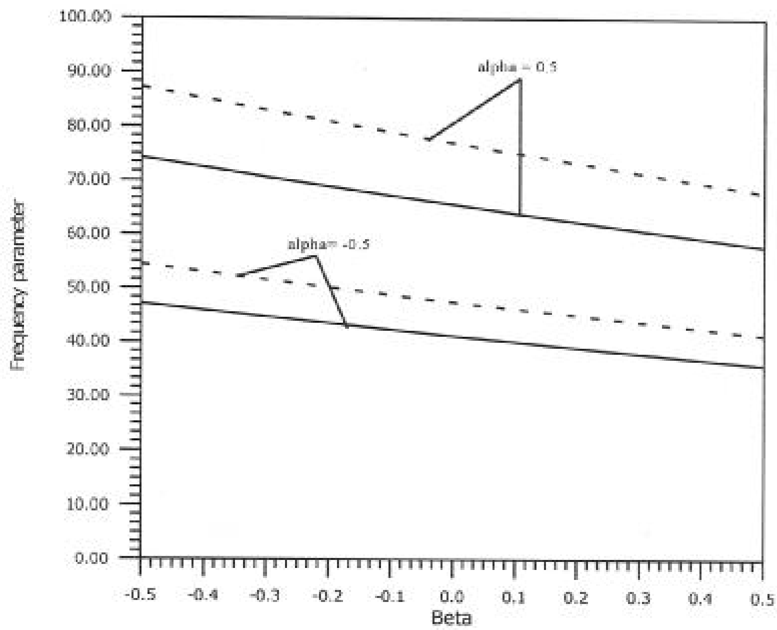

Fig.6. (*C-S* conditions) Variation of second mode of frequency parameter with density parameter. (Soild curve – Orthotropic plate, Dashed curve – Monoclinic plate).

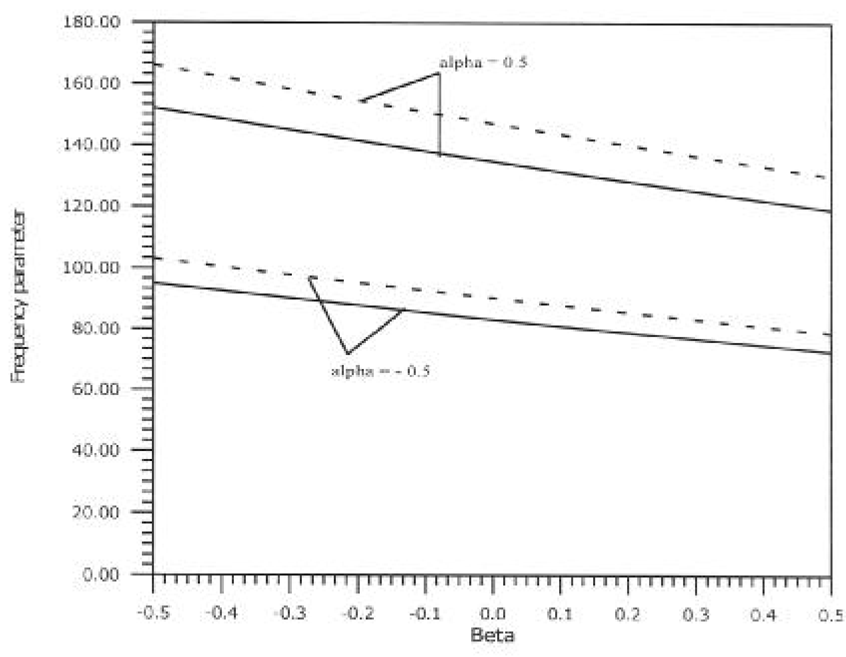

Fig.7. (*C-S* conditions) Variation of third mode of frequency parameter with density parameter. (Soild curve – Orthotropic plate, Dashed curve – Monoclinic plate).

Figure 8 depicts the variation of Ω with d when the plates are subjected to *C-S* conditions. At $d = 0.25$ with $\alpha = \beta = 0.5$, the values of Ω are *16.77*, *56.24*, *117.77* for a monoclinic plate and *16.14*, *55.97*, *117.06* in an orthotropic plate for the first, second, third modes of vibrations respectively. As the value of *d* increases through the values *0.25* to *2.0*, the values of frequency parameter Ω increase for the monoclinic as well as for orthotropic plates. Thus we conclude that the frequency of vibrations in the monoclinic plate is higher than that in the orthotropic plate.

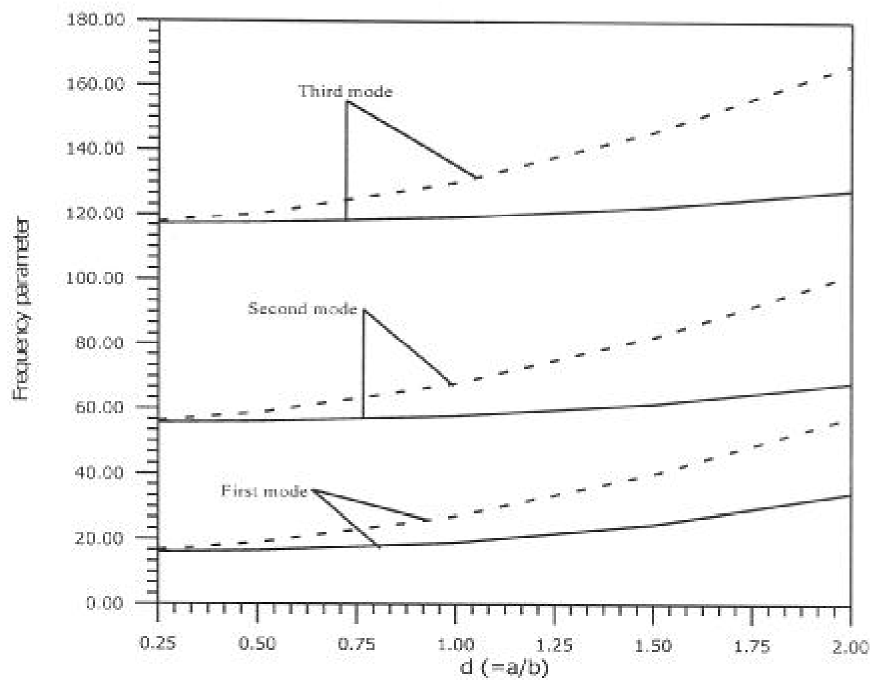

Fig.8. (*C-S* conditions) Variation of frequency parameters with aspect ratio. (Solid curve – orthotropic plate, Dashed curve – Monoclinic plate, $\alpha = \beta = 0.5$).

Figures 9, 10 and 11 depict the same what is presented in Figs 1, 2 and 3 for *C-F* conditions of the plates. Here we conclude that the frequency of vibrations in the monoclinic plate is higher than that in the orthotropic plate. The results at $\beta = -0.5$ are given as follows:

Modes		Monoclinic	Orthotropic
I	$(\alpha = -0.5)$	*8.54*	*7.50*
	$(\alpha = 0.5)$	*16.59*	*12.30*
II	$(\alpha = -0.5)$	*27.50*	*24.80*
	$(\alpha = 0.5)$	*41.83*	*36.22*
III	$(\alpha = -0.5)$	*63.00*	*59.00*
	$(\alpha = 0.5)$	*98.19*	*91.00*

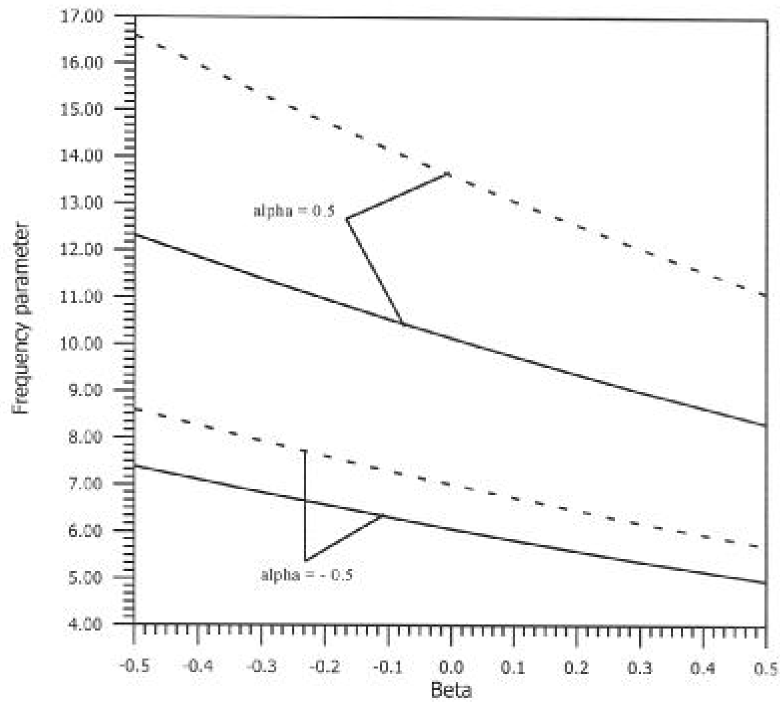

Fig.9. (*C-F* conditions) Variation of first mode of frequency parameter with density parameter. (Soild curve – Orthotropic plate, Dashed curve – Monoclinic plate).

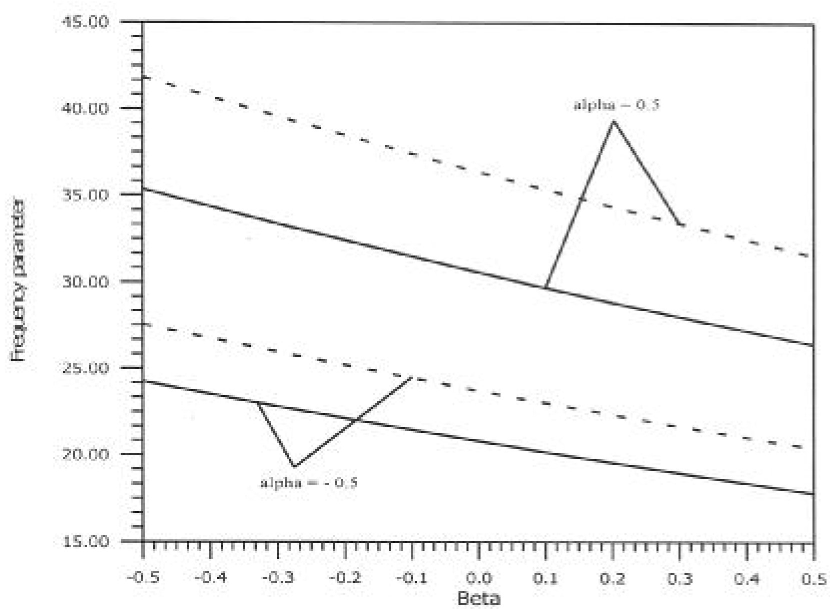

Fig.10. (*C-F* conditions) Variation of second mode of frequency parameter with density parameter. (Soild curve – Orthotropic plate, Dashed curve – Monoclinic plate).

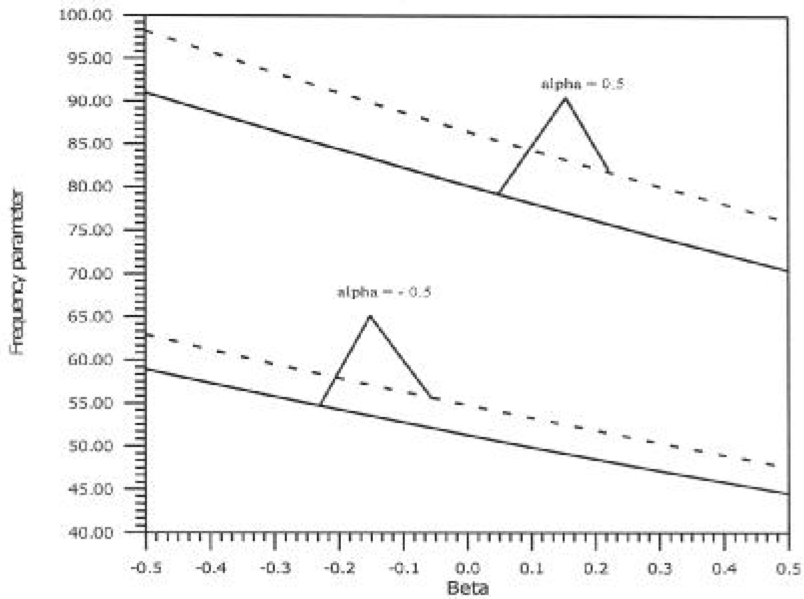

Fig.11. (*C-F* Conditions) Variation of third mode of frequency parameter with density parameter. (Soild curve – Orthotropic plate, Dashed curve – Monoclinic plate).

It can be concluded from Figs 5-7 and 9-11 that all the three modes of propagation decrease with the increase of density parameter in both types of plates.

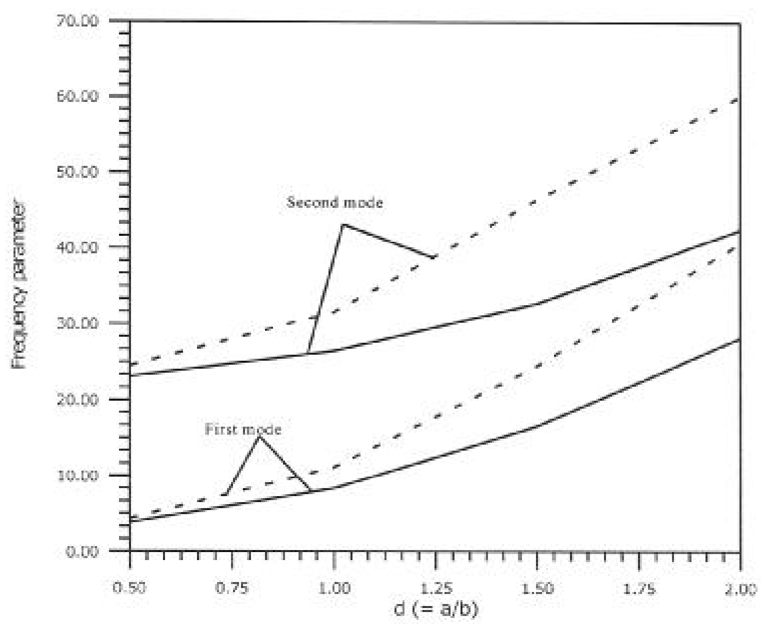

Fig.12. (*C-F* conditions) Variation of frequency parameters with aspect ratio. (Solid curve – orthotropic plate, Dashed curve – Monoclinic plate, $\alpha = \beta = 0.5$).

In Figure 12, we note that when $d = 0.5$, $(\alpha = 0.5, \beta = 0.5)$ and plates are subjected to *C-F* conditions, the value of Ω is *4.28* for a monoclinic plate and *3.77* for an orthotropic plate for the first mode of vibration; *24.49* for a monoclinic plate and *23.12* for an orthotropic plate for the second mode of vibration. As the value of ***d*** increases through the values *0.5* to *2.0*, the values of frequency parameter Ω increase for the monoclinic as well as for orthotropic plates. However, they increase much faster with ***d*** in the monoclinic plate than that in the orthotropic plate. Thus we conclude that the frequency of vibrations in the monoclinic plate is higher than that in the orthotropic plate.

Normalized displacements $W_{norm} = W/W_{\max}$ are shown in Figs 13-15 for first three modes of vibrations for ***C-C***, ***C-S*** and ***C-F*** conditions respectively.

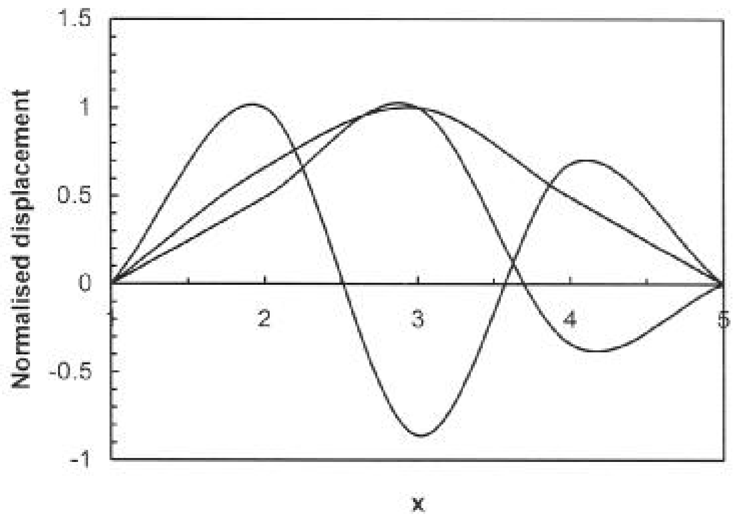

Fig.13. (*C-C* condition) Normalized displacements of monoclinic plate for the first three modes of vibrations.

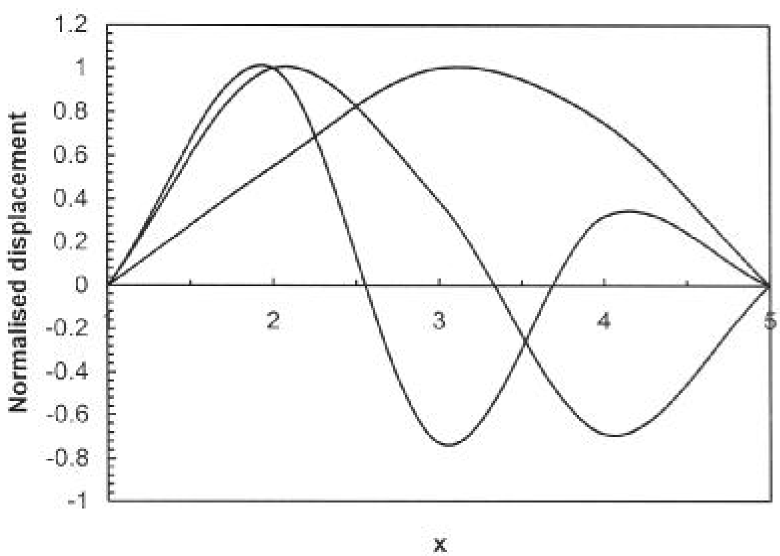

Fig.14. (*C-S* condition) Normalized displacements of monoclinic plate for the first three modes of vibrations.

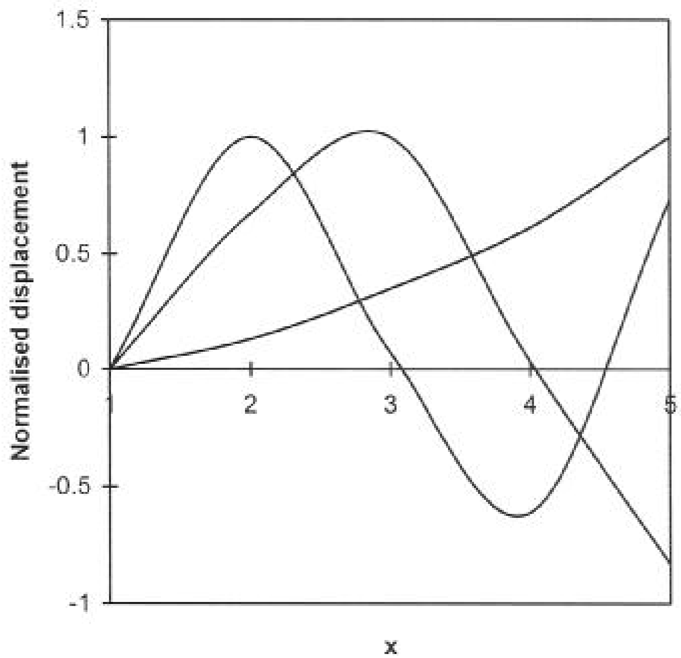

Fig.15. (*C-F* condition) Variation of normalized displacements with *X* in monoclinic plate for the first three modes of vibrations.

6. Conclusion

Free transverse vibrations of a rectangular plate composed of a monoclinic material with exponentially varying thickness and density have been studied. The equation of motion is developed using Hamilton's principle and solved by using Chebyshev polynomials. The frequency equations of the plate are derived with different boundary conditions namely, *C-S-C-S*, *C-S-S-S* and *C-S-F-S*, where *C*, *S* and *F* denote the clamped, simply supported and free edge respectively. These equations are then solved numerically for various combinations of physical parameters. The results show that the values of frequency parameter for the *C-C* plate are always greater than those for *C-S* plate and the values of frequency parameter for the *C-S* plate are greater than those for *C-F* plates for the same set of values of various plate parameters. The values of frequency parameter decrease with an increase of density parameter for first three modes of vibration and for all the three boundary conditions. However, the values of frequency parameter increase with an increase of the thickness parameter. A comparison of frequency curves for monoclinic and orthotropic plates show that the values of frequency parameter with respect to density parameter for a monoclinic plate are greater than those for an orthotropic plate in all the three modes of vibration. A similar behavior is observed with respect to thickness parameter with the observation that the values of frequency parameter increase faster in a monoclinic plate than that in an orthotropic plate for the first three modes of vibrations. No appreciable difference is observed in the normalized displacements of monoclinic and orthotropic plates in all the three possibilities considered.

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Nomenclature

a – length of the plate
b – breadth of the plate
C – clamped edge
c_{ij} – elastic constants
d – aspect ratio
e_{ij} – strain tensor
F – free edge
h – thickness of the plate
L – Lagrangian
p – positive integer
S – simply supported edge
T – kinetic energy
t_{ij} – stress tensor
u, v, w – Cartesian components of displacement
V – strain energy
x, y, z – Cartesian coordinates
α – taper constant
β – density parameter
ρ – density
ω – radial frequency

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