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A SUB-PARAMETRIC SHEAR DEFORMABLE ELEMENT FOR FREE VIBRATION ANALYSIS OF THICK/THIN RECTANGULAR PLATES WITH TAPERED THICKNESS

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A sub-parametric shear deformable element is proposed for free vibration analysis of isotropic plates with linearly varying thickness in one direction. The element has sixteen nodes and thirty-six degrees of freedom. The transverse displacement and bending rotations are taken as independent field variables. The polynomials used to express these variables are of the same order. The geometry of the element is defined by a polynomial of lower order than the polynomials used for field variables. The entire formulation is made based on first-order shear deformation theory (FSDT). The rotary inertia is included in the consistent mass matrix for the analysis. Isotropic plates with different thickness ratios (varying from *0.01* to *0.2*), tapered ratios, aspect ratios and boundary conditions are analyzed. The results obtained by the present element show an excellent agreement with the available published results. Some numerical results have been given as new results.

Key words: sub-parametric element, first order shear deformation theory, rotary inertia, tapered thickness, consistent mass matrix.

1. Introduction

Structures of plates have wide applications in ships, aircrafts, bridges, etc. A thorough dynamic study of their behavior and characteristics is essential to assess and use the full potentials of plates. The finite element method (FEM) is one of the most versatile analysis tool in engineering structures (Zienkiewick and Taylor, 1988; Wang *et al*., 2000). The plate bending is one of the first problems where finite element was successfully applied. A large number of plate bending elements are available in the literature (Batoz *et al*., 1980; Bhashyam and Gallagher, 1984; Cheung and Chen, 1989; Hrabok and Hrudey, 1984; Hughes and Tezduyaf, 1981; Hughes and Cohen, 1978; Petrolito, 1989; Pugh *et al*., 1987; Pryor *et al*., 1970; Rao *et al*., 1974; Salerno and Goldberg, 1968 and Yuan and Miller, 1989). But use of sub-parametric elements for the analysis of the plate is very rare. Such elements have been rarely applied. In one of these applications, Delpak (1967) used two- and three-dimensional geometry for defining linear shape functions. The variation of the unknown functions (i.e., displacements) was, however, defined by functions including nodal derivatives with respect to global co-ordinates. In another application, Ergatoudis (1966) studied an axis-symmetric shell problem.

In the aeronautical field, analysis of plates with variable thickness has been of great interest due to their utility in aircraft wings. Mizusawa (1993) have used the collocation method with orthogonal polynomials for vibration analysis of rectangular plates of variable thickness. The free vibrations of a wide range of tapered rectangular plates with an arbitrary number of intermediate line supports in one or two directions were invested by Cheung and Zhou (1999). A new set of admissible functions was developed for analysis of tapered plates. Zhou (2002) studied the free vibrations of point-supported rectangular plates with variable thickness using the Rayleigh-Ritz method. Pulmano and Gupta (1976) investigated the frequency

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parameters of a linear elastic rectangular plate of varying thickness in one direction using the finite strip method.

In the present paper, a new sub-parametric element is proposed for the free vibration analysis of thick/thin plates with varying thickness. Plates having various boundary conditions, thickness ratios ($t/b = 0.01,\ 0.1$ and 0.2) and aspect ratios (a/b) are analyzed using the proposed element. The results obtained are compared with those available in literature to show the performance of the element. In all the cases the results obtained are in close agreement with the published analytical solutions.

2. Finite element formulation

The formulation is based on the Reissner-Mindlin plate theory. In this theory it is assumed that the transverse deflection of the plate is small compared to the plate thickness and the normal to the plate mid surface which is taken as the reference plane remains straight but may not remain normal to the deformed mid surface. The proposed element is shown in Fig.1.

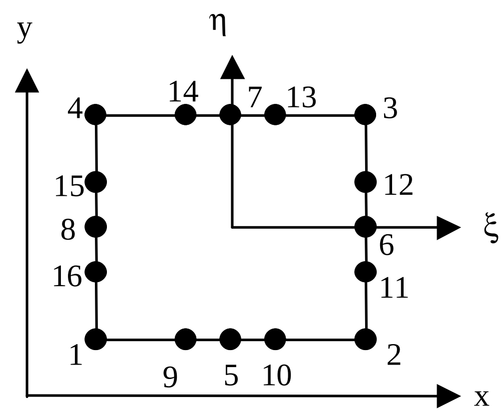

Fig.1. Element configuration.

The element has three degrees of freedom (w, θ_x, θ_y) at nodes 1-4 and 9-16. Nodes 1-4 are at the corner points, 5-8 are at midpoints of the sides of the element and 9-16 are at equidistant points from the mid side nodes. The natural co-ordinates of the nodes are $(-1,-1)$, $(1,-1)$, $(1,1)$, $(-1,1)$, $(0,-1)$, $(1,0)$, $(0,1)$, $(-1,0)$, $(-1/3,-1)$, $(1/3,-1)$, $(1,-1/3)$, $(1,1/3)$, $(1/3,1)$, $(-1/3,1)$, $(-1,1/3)$ and $(-1,-1/3)$. The co-ordinates of any point within the element with respect to the global co-ordinate system are given by

$$x = \sum_{i=1}^{8} N_i x_i \ , \qquad y = \sum N_i y_i \ . \tag{2.1}$$

The displacement component (w) and the rotations of the normal (θ_x and θ_y) are taken as the independent field variables and are expressed as follows

$$w = \sum_{i=1}^{4} N_{wi} w_i + \sum_{i=9}^{16} N_{wi} w_i \,,$$

$$\theta_x = \sum_{i=1}^{4} N_{wi} \theta_{xi} + \sum_{i=9}^{16} N_{wi} \theta_{xi} \,, \tag{2.2}$$

$$\theta_y = \sum_{i=1}^{4} N_{wi} \theta_{yi} + \sum_{i=9}^{16} N_{wi} \theta_{yi}$$

where N_i and N_{wi} are the shape functions which are obtained as follows

$$N_i = \frac{1}{4}(1+\xi_i\xi)(1+\eta_i\eta)(\xi_i\xi+\eta_i\eta-1) \quad \text{for} \quad i \le 4 \,,$$

$$N_i = \frac{1}{2}\left(1-\xi^2\right)(1+\eta_i\eta) \quad \text{for} \quad i = 5,7 \,,$$

$$N_i = \frac{1}{2}(1+\xi_i\xi)\left(1-\eta^2\right) \quad \text{for} \quad i = 6,8 \,,$$

$$N_{w9} = \frac{9}{32}(1-3\xi)\left(1-\xi^2\right)(1-\eta)\,, \qquad N_{w10} = \frac{9}{32}(1+3\xi)\left(1-\xi^2\right)(1-\eta),$$

$$N_{w11} = \frac{9}{32}(1+\xi)(1-3\eta)\left(1-\eta^2\right), \qquad N_{w12} = \frac{9}{32}(1+\xi)(1+3\eta)\left(1-\eta^2\right),$$

$$N_{w13} = \frac{9}{32}(1+3\xi)\left(1-\xi^2\right)(1+\eta), \qquad N_{w14} = \frac{9}{32}(1-3\xi)\left(1-\xi^2\right)(1+\eta), \tag{2.3}$$

$$N_{w15} = \frac{9}{32}(1-\xi)(1+3\eta)\left(1-\eta^2\right), \qquad N_{w16} = \frac{9}{32}(1-\xi)(1-3\eta)\left(1-\eta^2\right),$$

$$N_{w1} = N_1 - \frac{2}{3}(N_{w9} + N_{w16}) - \frac{1}{3}(N_{w10} + N_{w15})\,,$$

$$N_{w2} = N_2 - \frac{2}{3}(N_{w10} + N_{w11}) - \frac{1}{3}(N_{w9} + N_{w12})\,,$$

$$N_{w3} = N_3 - \frac{2}{3}(N_{w12} + N_{w13}) - \frac{1}{3}(N_{w11} + N_{w14})\,,$$

$$N_{w4} = N_4 - \frac{2}{3}(N_{w14} + N_{w15}) - \frac{1}{3}(N_{w13} + N_{w16})\,.$$

Equation (2.2) can be expressed in a matrix form as below

$$\begin{Bmatrix} w \\ \theta_x \\ \theta_y \end{Bmatrix} = \left[\overline{N}\right]\{\delta\} \tag{2.4}$$

where

$$\{\delta\}^T = \left\{w_1\theta_{x1}\theta_{y1}........w_4\theta_4\theta_4 \middle| w_9\theta_{x9}\theta_{y9}......w_{16}\theta_{x16}\theta_{y16}\right\},$$

$$\left[\overline{N}\right] = [[\beta_1]...[\beta_4][\beta_9].....[\beta_{16}]] \quad \text{in which}$$

$$[\beta_i] = \begin{bmatrix} N_{wi} & 0 & 0 \\ 0 & N_{wi} & 0 \\ 0 & 0 & N_{wi} \end{bmatrix} \quad \text{for} \quad i = 1,...,4,9,....,16\,. \tag{2.5}$$

As rotations of the normal θ_x and θ_y are independent variables and they are not derivatives of w, the effect of shear deformation can be easily incorporated as

$$\begin{Bmatrix} \phi_x \\ \phi_y \end{Bmatrix} = \begin{Bmatrix} \theta_x - \dfrac{\partial w}{\partial x} \\ \theta_y - \dfrac{\partial w}{\partial y} \end{Bmatrix} \tag{2.6}$$

where ϕ_x and ϕ_y are average shear strain over the entire plate thickness and θ_x and θ_y are the total rotations of the normal.

The generalized stress-strain relationship may be expressed as

$$\{\sigma\} = [D]\{\varepsilon\}. \tag{2.7}$$

In the above equation the generalized stress vector is

$$\{\sigma\}^T = \{M_x \quad M_y \quad M_{xy} \quad Q_x \quad Q_y\}. \tag{2.8}$$

The generalized strain vector $\{\varepsilon\}$ in terms of displacement fields is

$$\{\varepsilon\} = \begin{Bmatrix} -\partial\theta_x/\partial x \\ -\partial\theta_y/\partial y \\ -\partial\theta_x/\partial y - \partial\theta_y/\partial x \\ \partial w/\partial x - \theta_x \\ \partial w/\partial y - \theta_y \end{Bmatrix}, \tag{2.9}$$

and the rigidity matrix $[D]$ is given by

$$[D]=\frac{Eh_0^3(y\delta+1)^3}{12b^3(1-\nu^2)}\begin{bmatrix} 1 & \nu & 0 & 0 & 0 \\ \nu & 1 & 0 & 0 & 0 \\ 0 & 0 & \frac{1-\nu}{2} & 0 & 0 \\ 0 & 0 & 0 & \frac{b^2(1-\nu)}{2kh_0^2(y\delta+1)^2} & 0 \\ 0 & 0 & 0 & 0 & \frac{b^2(1-\nu)}{2kh_0^2(y\delta+1)^2} \end{bmatrix},\ k=\pi^2/12,\ \delta=(h_1-h_0)/h_0\ ,$$

y = ordinate of the Gauss point (see Fig.2).

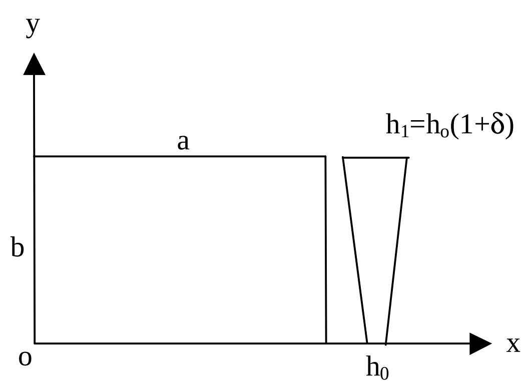

Fig.2. Geometry of rectangular plates.

Substituting Eq.(2.2) in Eq.(2.9) the strain-displacement relationship may be expressed as

$$\{\varepsilon\}=[B]\{\delta\}$$

where $[B]$ is a (5×36) matrix.

Thus, the element stiffness matrix $[K]^e$ is given by

$$[K]^e=\int[B]^T[D][B]|J|d\xi\, d\eta\,. \tag{2.10}$$

In a similar manner, the consistent mass matrix can be expressed as

$$[M]^e=\int[\overline{N}]^T[\overline{\rho}][\overline{N}]|J|d\xi\, d\eta \tag{2.11}$$

where $$\left[\bar{\rho}\right]=\begin{bmatrix}\rho t_g & 0 & 0\\ 0 & \rho t_g^3/12 & 0\\ 0 & 0 & \rho t_g^3/12\end{bmatrix}, \qquad t_g=h_0(y\delta+1)/b\ ,$$

y = ordinate of the Gauss point (see Fig.2).

Integration of the above Eqs (2.11) and (2.12) is carried out using the Gauss Quadrature Method.

The element stiffness and consistent mass matrices having an order of thirty-six have been evaluated for all the elements and they have been assembled together to form the overall stiffness $[K]$ and mass matrix $[M]$ respectively. The storage of $[K]$ and $[M]$ has been done in single array following the skyline storage technique. Once $[K]$ and $[M]$ are obtained, the equation of motion of the plate may be expressed as

$$[K]-\omega^2[M]=0\ . \tag{2.12}$$

After incorporating the boundary conditions in the above equation it has been solved by simultaneous iterative technique of Corr and Jennings (1976) to get frequency ω for first few modes.

3. Numerical examples

A number of isotropic rectangular plates (Fig.2) with different boundary conditions (*SSCF*, *SSSS*, *SSCC*, *CCCC* and *FFFC*), thickness ratios (h_0/b), tapered ratios (δ) and aspect ratios (a/b) have been analysed using the proposed element. The study has been made from thin $(h_0/b=0.01)$ to sufficiently thick $(h_0/b=0.2)$ plates. The convergence study of the element is shown in Tab.1. The first six frequencies obtained by the present analysis have been given in non-dimensional forms in Tabs 1-3 with those of Mizusawa (1993). Mizusawa (1993) has analysed the problems using spline finite strip method considering Mindlin's plate theory. In all examples, the results are given with 12×12 mesh divisions. The first six non-dimensional frequency parameters for clamped and cantilever plates have been displayed in Tabs 4-5 as new results. From the results it is seen that the performance of the present element is good for thick plates $(h_0/b=0.2)$ as well as for thin plates $(h_0/b=0.01)$. The boundary conditions of a plate (Fig.2) with simply supported at edges $x=0$ and $x=a$, clamped at edge $y=0$ and free at edge $y=b$ have been symbolised as *SSCF*. Poisson's ratio for the material is taken as 0.3.

4. Conclusions

A shear deformable sub-parametric element is proposed for free vibration analysis of rectangular plates with tapered thickness. The analysis has been performed for thin to very thick plates with different aspect ratios, tapered ratios and boundary conditions. The entire formulation has been made based on first order shear deformation theory. There is no shear-locking problem even for thin plates. Rotary inertia has been taken into account in the present formulation. From Tab.1 it is clear that the convergence of the proposed element is very fast. From the examples it is concluded that the results obtained by the proposed element have an excellent agreement with the available solutions and the sub-parametric finite element formulation is equally effective as the isoparametric finite element formulation.

Table 1. Convergence study of frequency parameters, $\lambda = \omega a^2 \sqrt{(\rho h_0 / D_0)}$ for rectangular plates: Boundary Conditions (*SSCF*).

Thickness ratio (h_0/b)	Tapered ratio (δ)	Aspect ratio (a/b)	Sources	Mode sequences					
				1	2	3	4	5	6
0.01	0.25	0.5	PS-4	12.296	18.370	30.918	46.819	48.694	53.223
			PS-8	12.213	17.784	28.951	45.443	46.264	51.282
			PS-12#	12.211	17.769	28.902	45.321	46.217	51.161
			Ref.$	12.212	17.776	28.915	45.335	46.213	51.159
		2.0	PS-4	25.287	63.623	112.28	130.61	177.29	267.67
			PS-8	24.651	58.284	108.91	115.64	148.23	196.92
			PS-12	24.643	58.185	108.84	115.19	147.24	195.52
			Ref.$	24.654	58.195	108.87	115.17	147.24	195.40
	1.00	0.5	PS-4	17.336	24.187	39.374	61.736	65.090	74.274
			PS-8	17.202	23.781	37.984	59.111	59.458	71.893
			PS-12	17.198	23.766	37.941	59.014	59.278	71.731
			Ref.$	17.198	23.771	37.952	59.027	59.251	71.709
		2.0	PS-4	31.306	85.216	137.68	177.60	217.10	329.59
			PS-8	30.825	80.637	134.86	163.54	191.65	276.53
			PS-12	30.821	80.525	134.81	162.81	190.78	272.27
			Ref.$	30.833	80.527	134.85	162.76	190.78	271.96
0.1	0.25	0.5	PS-4	11.050	15.329	23.287	33.911	35.155	37.928
			PS-8	11.049	15.322	23.242	33.704	35.135	37.867
			PS-12	11.049	15.322	23.242	33.700	35.134	37.866
			Ref.$	11.049	15.321	23.239	33.695	35.130	37.860
		2.0	PS-4	23.855	55.675	100.42	108.46	133.88	181.58
			PS-8	23.850	55.676	100.36	107.72	132.90	176.86
			PS-12	23.850	55.675	100.36	107.71	132.89	176.80
			Ref.$	23.850	55.673	100.36	107.71	132.88	176.78
	1.00	0.5	PS-4	14.441	18.752	27.416	38.832	40.847	44.647
			PS-8	14.440	18.747	27.374	38.574	40.782	44.561
			PS-12	14.440	18.746	27.373	38.570	40.781	44.560
			Ref.$	14.438	18.744	27.369	38.563	40.774	44.550
		2.0	PS-4	29.077	74.676	118.20	145.29	162.55	228.34
			PS-8	29.075	74.570	118.16	144.30	161.73	223.45
			PS-12	29.074	74.568	118.16	144.29	161.72	223.40
			Ref.$	29.073	74.565	118.15	144.27	161.71	223.37
0.2	0.25	0.5	PS-4	9.0406	11.851	16.715	22.702	23.959	25.509
			PS-8	9.0405	11.849	16.697	22.613	23.951	25.482
			PS-12	9.0405	11.849	16.697	22.612	23.951	25.482
			Ref.$	9.0392	11.847	16.694	22.606	23.944	25.475
		2.0	PS-4	22.397	50.627	84.389	93.527	109.50	146.77
			PS-8	22.395	50.596	84.373	93.196	109.24	144.69
			PS-12	22.395	50.596	84.373	93.191	109.24	144.65
			Ref.$	22.394	50.592	84.363	93.181	109.22	144.63
	1.00	0.5	PS-4	10.745	13.277	17.985	23.118	25.952	27.289
			PS-8	10.745	13.275	17.970	23.063	25.942	27.251
			PS-12	10.745	13.275	17.970	23.062	25.939	27.251
			Ref.$	10.743	13.272	17.965	23.056	25.931	27.242
		2.0	PS-4	26.234	63.761	92.108	115.13	122.98	165.88
			PS-8	26.233	63.727	92.094	114.75	122.74	164.24
			PS-12	26.233	63.726	92.094	114.74	122.74	164.23
			Ref.$	26.231	63.718	92.080	114.72	122.72	164.19

PS-12 represents present solution with 12×12 mesh division; Ref.$ - Mizusawa (1993)

Table 2. Frequency parameters, $\lambda = \omega a^2 \sqrt{(\rho h_0 / D_0)}$ for rectangular plates: Boundary Conditions (*SSSS*).

Thickness ratio (h_0/b)	Tapered ratio (δ)	Aspect ratio (a/b)	Sources	Mode sequences 1	2	3	4	5	6
0.01	*0.25*	*0.5*	PS-12	*13.817*	*22.129*	*35.899*	*46.531*	*55.130*	*55.258*
			Ref.[$]	*13.817*	*22.127*	*35.894*	*46.530*	*55.118*	*55.247*
		1.0	PS-12	*22.165*	*55.270*	*55.336*	*88.550*	*110.08*	*110.50*
			Ref.[$]	*22.164*	*55.266*	*55.332*	*88.508*	*110.07*	*110.49*
		2.0	PS-12	*55.371*	*88.681*	*144.01*	*188.14*	*221.28*	*221.49*
			Ref.[$]	*55.368*	*88.656*	*143.92*	*188.13*	*221.06*	*221.33*
	0.50	*0.5*	PS-12	*15.232*	*24.479*	*39.651*	*50.542*	*60.830*	*61.229*
			Ref.[$]	*15.231*	*24.478*	*39.646*	*50.538*	*60.817*	*61.216*
		1.0	PS-12	*24.543*	*60.930*	*61.164*	*97.951*	*120.53*	*122.00*
			Ref.[$]	*24.543*	*60.925*	*61.160*	*97.911*	*120.50*	*121.99*
		2.0	PS-12	*61.187*	*98.195*	*159.20*	*207.89*	*243.95*	*244.79*
			Ref.[$]	*61.185*	*98.171*	*159.11*	*207.88*	*243.70*	*244.64*
	1.00	*0.5*	PS-12	*17.905*	*29.035*	*46.846*	*57.424*	*71.687*	*72.616*
			Ref.[$]	*17.904*	*29.033*	*46.841*	*57.416*	*71.673*	*72.597*
		1.0	PS-12	*29.185*	*71.623*	*72.392*	*116.17*	*139.29*	*143.96*
			Ref.[$]	*29.184*	*71.616*	*72.388*	*116.13*	*139.24*	*143.95*
		2.0	PS-12	*72.361*	*116.76*	*188.51*	*245.84*	*286.80*	*289.70*
			Ref.[$]	*72.359*	*116.74*	*188.40*	*245.83*	*286.46*	*289.55*
0.1	*0.25*	*0.5*	PS-12	*12.506*	*19.052*	*28.802*	*35.717*	*40.755*	*40.796*
			Ref.[$]	*12.506*	*19.050*	*28.799*	*35.712*	*40.748*	*40.790*
		1.0	PS-12	*21.223*	*50.025*	*50.065*	*76.208*	*92.179*	*92.374*
			Ref.[$]	*21.223*	*50.023*	*50.063*	*76.201*	*92.170*	*92.364*
		2.0	PS-12	*53.854*	*84.893*	*134.48*	*171.53*	*200.11*	*200.26*
			Ref.[$]	*53.853*	*84.891*	*134.47*	*171.52*	*200.09*	*200.25*
	0.50	*0.5*	PS-12	*13.538*	*20.507*	*30.682*	*37.602*	*43.003*	*43.125*
			Ref.[$]	*13.537*	*20.505*	*30.678*	*37.596*	*42.996*	*43.117*
		1.0	PS-12	*23.282*	*54.153*	*54.284*	*82.027*	*98.303*	*98.909*
			Ref.[$]	*23.281*	*54.149*	*54.280*	*82.019*	*98.291*	*98.897*
		2.0	PS-12	*59.142*	*93.126*	*146.65*	*187.46*	*216.62*	*217.14*
			Ref.[$]	*59.140*	*93.123*	*146.65*	*187.45*	*216.60*	*217.12*
	1.00	*0.5*	PS-12	*15.347*	*23.006*	*33.777*	*40.542*	*46.568*	*46.844*
			Ref.[$]	*15.345*	*23.003*	*33.771*	*40.535*	*46.558*	*46.834*
		1.0	PS-12	*27.119*	*61.387*	*61.742*	*92.023*	*108.35*	*109.88*
			Ref.[$]	*27.118*	*61.382*	*61.737*	*92.011*	*108.33*	*109.86*
		2.0	PS-12	*68.965*	*108.48*	*168.76*	*214.08*	*245.56*	*246.97*
			Ref.[$]	*68.963*	*108.47*	*168.74*	*214.06*	*245.53*	*246.95*
0.2	*0.25*	*0.5*	PS-12	*10.188*	*14.552*	*20.502*	*24.506*	*27.251*	*27.259*
			Ref.[$]	*10.186*	*14.549*	*20.497*	*24.500*	*27.243*	*27.252*
		1.0	PS-12	*19.055*	*40.752*	*40.765*	*58.207*	*68.292*	*68.340*
			Ref.[$]	*19.053*	*40.746*	*40.759*	*58.196*	*68.278*	*68.326*
		2.0	PS-12	*50.038*	*76.220*	*115.24*	*143.37*	*163.01*	*163.06*
			Ref.[$]	*50.036*	*76.213*	*115.22*	*143.35*	*162.98*	*163.04*
	0.50	*0.5*	PS-12	*10.751*	*15.207*	*21.204*	*25.187*	*27.959*	*27.984*
			Ref.[$]	*10.749*	*15.204*	*21.199*	*25.180*	*27.951*	*27.976*
		1.0	PS-12	*20.516*	*43.002*	*43.039*	*60.829*	*70.931*	*71.057*
			Ref.[$]	*20.514*	*42.995*	*43.032*	*60.817*	*70.916*	*71.041*
		2.0	PS-12	*54.177*	*82.064*	*122.83*	*151.90*	*172.01*	*172.16*
			Ref.[$]	*54.173*	*82.056*	*122.81*	*151.88*	*171.98*	*172.13*
	1.00	*0.5*	PS-12	*11.650*	*16.212*	*22.234*	*26.175*	*27.399*	*28.974*
			Ref.[$]	*11.648*	*16.208*	*22.228*	*26.167*	*27.390*	*28.965*
		1.0	PS-12	*23.027*	*46.599*	*46.667*	*64.850*	*74.914*	*75.129*
			Ref.[$]	*23.025*	*46.590*	*46.658*	*64.834*	*74.895*	*75.108*
		2.0	PS-12	*61.327*	*92.109*	*135.43*	*165.71*	*186.40*	*186.67*
			Ref.[$]	*61.321*	*92.098*	*135.41*	*165.68*	*186.36*	*186.63*

Table 3. Frequency parameters, $\lambda = \omega a^2 \sqrt{(\rho h_0 / D_0)}$ for rectangular plates: Boundary Conditions (*SSCC*).

Thickness ratio (h_0/b)	Tapered ratio (δ)	Aspect ratio (a/b)	Sources	Mode sequences					
				1	*2*	*3*	*4*	*5*	*6*
0.01	*0.25*	*0.5*	PS-12	*15.323*	*26.458*	*43.215*	*47.368*	*57.796*	*65.359*
			Ref.$	*15.322*	*26.453*	*43.203*	*47.335*	*57.749*	*65.334*
		1.0	PS-12	*32.444*	*61.318*	*77.587*	*105.93*	*114.25*	*144.22*
			Ref.$	*32.441*	*61.287*	*77.577*	*105.81*	*114.14*	*144.20*
		2.0	PS-12	*106.76*	*129.87*	*175.60*	*246.28*	*284.43*	*310.65*
			Ref.$	*106.76*	*129.76*	*175.16*	*245.41*	*284.41*	*310.31*
	0.50	*0.5*	PS-12	*16.895*	*29.204*	*47.658*	*51.720*	*63.926*	*72.026*
			Ref.$	*16.894*	*29.199*	*47.647*	*51.703*	*63.880*	*72.004*
		1.0	PS-12	*35.813*	*67.607*	*85.592*	*116.92*	*125.50*	*159.02*
			Ref.$	*35.810*	*67.575*	*85.582*	*116.80*	*125.37*	*158.99*
		2.0	PS-12	*117.84*	*143.34*	*193.72*	*271.42*	*313.75*	*342.65*
			Ref.$	*117.83*	*143.24*	*193.30*	*270.30*	*313.72*	*342.33*
	1.00	*0.5*	PS-12	*19.866*	*34.443*	*56.090*	*59.396*	*75.676*	*84.629*
			Ref.$	*19.865*	*34.438*	*56.079*	*59.369*	*75.625*	*84.606*
		1.0	PS-12	*42.234*	*79.492*	*100.79*	*137.86*	*146.15*	*187.01*
			Ref.$	*42.231*	*79.458*	*100.78*	*137.75*	*145.98*	*186.99*
		2.0	PS-12	*138.92*	*169.02*	*228.24*	*319.01*	*369.36*	*403.41*
			Ref.$	*138.91*	*168.92*	*227.83*	*317.83*	*369.33*	*403.12*
0.1	*0.25*	*0.5*	PS-12	*13.440*	*21.290*	*31.727*	*35.979*	*41.527*	*43.773*
			Ref.$	*13.439*	*21.287*	*31.722*	*35.974*	*41.520*	*43.763*
		1.0	PS-12	*29.337*	*53.760*	*64.155*	*85.162*	*94.031*	*108.38*
			Ref.$	*29.335*	*53.755*	*64.148*	*85.150*	*94.017*	*108.36*
		2.0	PS-12	*97.417*	*117.35*	*156.61*	*215.05*	*237.05*	*256.63*
			Ref.$	*97.411*	*117.34*	*156.59*	*215.02*	*237.02*	*256.59*
	0.50	*0.5*	PS-12	*14.490*	*22.698*	*33.456*	*37.911*	*43.799*	*45.760*
			Ref.$	*14.489*	*22.695*	*33.450*	*37.904*	*43.791*	*45.749*
		1.0	PS-12	*31.777*	*57.960*	*68.513*	*90.795*	*100.27*	*114.40*
			Ref.$	*31.775*	*57.955*	*68.504*	*90.782*	*100.25*	*114.37*
		2.0	PS-12	*105.62*	*127.11*	*169.40*	*231.85*	*253.26*	*274.06*
			Ref.$	*105.61*	*127.10*	*169.39*	*231.82*	*253.23*	*274.02*
	1.00	*0.5*	PS-12	*16.310*	*25.042*	*36.208*	*40.935*	*47.428*	*48.832*
			Ref.$	*16.308*	*25.038*	*36.201*	*40.927*	*47.417*	*48.819*
		1.0	PS-12	*36.066*	*65.238*	*75.702*	*100.17*	*110.54*	*123.90*
			Ref.$	*36.063*	*65.231*	*75.690*	*100.15*	*110.52*	*123.87*
		2.0	PS-12	*119.99*	*144.27*	*191.89*	*260.97*	*279.87*	*302.82*
			Ref.$	*119.98*	*144.25*	*191.86*	*260.93*	*279.83*	*302.76*
0.2	*0.25*	*0.5*	PS-12	*10.579*	*15.290*	*21.204*	*24.566*	*27.416*	*27.738*
			Ref.$	*10.577*	*15.287*	*21.198*	*24.559*	*27.408*	*27.729*
		1.0	PS-12	*23.884*	*42.315*	*46.628*	*61.160*	*68.854*	*72.854*
			Ref.$	*23.880*	*42.308*	*46.617*	*61.146*	*68.837*	*72.832*
		2.0	PS-12	*79.761*	*95.535*	*126.30*	*169.27*	*172.39*	*186.51*
			Ref.$	*79.748*	*95.520*	*126.28*	*169.23*	*172.35*	*186.47*
	0.50	*0.5*	PS-12	*11.113*	*15.854*	*21.783*	*25.245*	*28.116*	*28.322*
			Ref.$	*11.111*	*15.850*	*21.777*	*25.238*	*28.108*	*28.312*
		1.0	PS-12	*25.215*	*44.452*	*48.340*	*63.416*	*71.460*	*74.843*
			Ref.$	*25.210*	*44.444*	*48.328*	*63.401*	*71.441*	*74.819*
		2.0	PS-12	*84.126*	*100.86*	*133.26*	*177.81*	*178.48*	*193.36*
			Ref.$	*84.110*	*100.84*	*133.23*	*177.77*	*178.44*	*193.31*
	1.00	*0.5*	PS-12	*11.960*	*16.709*	*22.634*	*26.232*	*29.115*	*29.143*
			Ref.$	*11.958*	*16.704*	*22.627*	*26.225*	*29.106*	*29.132*
		1.0	PS-12	*27.358*	*47.840*	*50.906*	*66.836*	*75.388*	*77.780*
			Ref.$	*27.353*	*47.830*	*50.892*	*66.828*	*75.367*	*77.754*
		2.0	PS-12	*90.946*	*109.43*	*144.53*	*187.36*	*191.37*	*203.63*
			Ref.$	*90.924*	*109.41*	*144.50*	*187.31*	*191.32*	*203.57*

Table 4. Frequency parameters, $\lambda = \omega a^2 \sqrt{(\rho h_0 / D_0)}$ for rectangular plates: Boundary Conditions (*CCCC*).

Thickness ratio (h_0/b)	Tapered ratio (δ)	Aspect ratio (a/b)	Sources	Mode sequences 1	2	3	4	5	6
0.01	*0.25*	*0.50*	PS-12	*27.390*	*35.590*	*49.990*	*70.462*	*70.597*	*79.279*
		0.75	PS-12	*32.126*	*53.914*	*75.241*	*89.639*	*95.715*	*129.64*
		1.00	PS-12	*40.327*	*82.094*	*82.189*	*121.35*	*146.80*	*147.68*
		1.25	PS-12	*52.355*	*91.946*	*119.89*	*156.03*	*156.69*	*218.61*
		1.50	PS-12	*68.129*	*105.27*	*166.62*	*167.89*	*201.64*	*254.10*
		2.00	PS-12	*110.24*	*143.01*	*201.53*	*285.43*	*286.67*	*319.43*
	0.50	*0.50*	PS-12	*30.040*	*39.356*	*55.173*	*76.162*	*77.812*	*87.737*
		0.75	PS-12	*35.417*	*59.518*	*82.408*	*98.848*	*105.82*	*143.05*
		1.00	PS-12	*44.505*	*90.342*	*90.683*	*133.99*	*160.83*	*162.67*
		1.25	PS-12	*57.788*	*101.36*	*132.25*	*171.49*	*172.89*	*241.31*
		1.50	PS-12	*75.198*	*116.13*	*183.72*	*184.94*	*222.41*	*279.03*
		2.00	PS-12	*121.67*	*157.79*	*222.19*	*314.26*	*316.19*	*352.21*
	1.00	*0.50*	PS-12	*34.859*	*46.624*	*65.091*	*85.824*	*91.500*	*103.42*
		0.75	PS-12	*41.636*	*70.232*	*95.320*	*116.33*	*125.30*	*168.74*
		1.00	PS-12	*52.461*	*105.72*	*106.85*	*158.25*	*185.83*	*191.25*
		1.25	PS-12	*68.148*	*119.16*	*155.74*	*200.04*	*203.86*	*284.95*
		1.50	PS-12	*88.674*	*136.77*	*215.89*	*217.23*	*261.99*	*324.82*
		2.00	PS-12	*143.45*	*186.00*	*261.47*	*368.59*	*372.19*	*414.59*
0.1	*0.25*	*0.50*	PS-12	*20.541*	*25.802*	*34.411*	*43.263*	*45.385*	*47.413*
		0.75	PS-12	*27.181*	*42.870*	*56.105*	*65.694*	*68.463*	*87.503*
		1.00	PS-12	*35.647*	*67.097*	*67.127*	*93.412*	*109.33*	*110.45*
		1.25	PS-12	*47.067*	*78.706*	*98.884*	*124.63*	*125.26*	*164.94*
		1.50	PS-12	*61.694*	*92.321*	*138.07*	*140.61*	*162.99*	*201.47*
		2.00	PS-12	*100.40*	*128.03*	*175.90*	*238.45*	*240.95*	*261.89*
	0.50	*0.50*	PS-12	*21.649*	*27.187*	*36.070*	*44.698*	*47.293*	*49.142*
		0.75	PS-12	*29.137*	*45.611*	*59.065*	*69.216*	*72.056*	*91.612*
		1.00	PS-12	*38.483*	*71.528*	*71.610*	*99.080*	*115.20*	*116.46*
		1.25	PS-12	*50.944*	*84.527*	*105.58*	*132.66*	*133.18*	*174.68*
		1.50	PS-12	*66.841*	*99.558*	*147.46*	*150.51*	*173.78*	*213.87*
		2.00	PS-12	*108.85*	*138.54*	*189.64*	*254.75*	*258.40*	*279.57*
	1.00	*0.50*	PS-12	*23.396*	*29.360*	*38.622*	*46.811*	*50.178*	*51.689*
		0.75	PS-12	*32.434*	*50.076*	*63.702*	*74.739*	*77.679*	*97.937*
		1.00	PS-12	*43.404*	*78.821*	*78.977*	*108.24*	*124.39*	*125.94*
		1.25	PS-12	*57.738*	*94.415*	*116.58*	*145.82*	*146.10*	*190.34*
		1.50	PS-12	*75.882*	*112.07*	*162.89*	*167.10*	*191.53*	*233.95*
		2.00	PS-12	*123.68*	*156.95*	*213.40*	*281.50*	*287.90*	*308.70*
0.2	*0.25*	*0.50*	PS-12	*13.701*	*17.019*	*22.076*	*25.758*	*28.163*	*28.349*
		0.75	PS-12	*20.317*	*30.207*	*37.248*	*43.565*	*44.818*	*55.736*
		1.00	PS-12	*28.148*	*48.277*	*48.281*	*64.429*	*73.042*	*74.079*
		1.25	PS-12	*38.007*	*59.701*	*71.581*	*88.244*	*88.955*	*112.19*
		1.50	PS-12	*50.255*	*72.303*	*100.19*	*104.36*	*116.87*	*141.98*
		2.00	PS-12	*82.227*	*103.19*	*137.67*	*173.30*	*181.56*	*189.56*
	0.50	*0.50*	PS-12	*14.047*	*17.431*	*22.551*	*26.095*	*28.674*	*28.757*
		0.75	PS-12	*21.143*	*31.173*	*38.157*	*44.669*	*45.886*	*56.910*
		1.00	PS-12	*29.544*	*49.945*	*49.966*	*66.365*	*74.963*	*76.044*
		1.25	PS-12	*40.040*	*62.268*	*74.103*	*91.191*	*91.952*	*115.19*
		1.50	PS-12	*53.010*	*75.807*	*103.73*	*108.54*	*120.96*	*146.63*
		2.00	PS-12	*86.768*	*108.72*	*144.37*	*179.40*	*189.28*	*196.35*
	1.00	*0.50*	PS-12	*14.542*	*18.009*	*23.230*	*26.561*	*29.317*	*29.370*
		0.75	PS-12	*22.384*	*32.562*	*39.458*	*46.272*	*47.410*	*58.585*
		1.00	PS-12	*31.707*	*52.371*	*52.456*	*69.175*	*77.791*	*78.936*
		1.25	PS-12	*43.233*	*66.170*	*77.779*	*95.517*	*96.438*	*119.61*
		1.50	PS-12	*57.353*	*81.241*	*108.88*	*114.89*	*126.99*	*153.57*
		2.00	PS-12	*93.901*	*117.46*	*154.85*	*188.25*	*201.13*	*206.40*

Table 5. Frequency parameters, $\lambda = \omega a^2 \sqrt{(\rho h_0 / D_0)}$ for rectangular plates: Boundary Conditions (*FFFC*).

Thickness ratio (h_0/b)	Tapered ratio (δ)	Aspect ratio (a/b)	Sources	Mode sequences 1	2	3	4	5	6
0.01	0.25	0.50	PS-12	1.1012	4.4135	6.2626	13.418	17.073	25.012
		0.75	PS-12	2.4927	7.0904	14.110	22.955	26.876	38.952
		1.00	PS-12	4.4484	10.084	24.781	30.145	34.890	60.301
		1.25	PS-12	6.9689	13.406	32.388	40.327	49.454	72.884
		1.50	PS-12	10.054	17.088	36.749	57.577	66.186	77.654
		2.00	PS-12	17.918	25.687	46.626	84.476	102.05	113.50
	0.50	0.50	PS-12	1.3495	5.1337	7.1523	14.814	19.066	26.681
		0.75	PS-12	3.0554	8.2156	16.112	25.426	29.097	43.496
		1.00	PS-12	5.4538	11.667	28.182	33.152	38.846	66.417
		1.25	PS-12	8.5449	15.527	35.928	45.914	55.302	78.501
		1.50	PS-12	12.329	19.849	40.997	65.702	73.900	84.700
		2.00	PS-12	21.973	30.095	52.609	92.727	116.46	128.19
	1.00	0.50	PS-12	1.8626	6.5420	8.8964	17.500	22.904	29.822
		0.75	PS-12	4.2183	10.428	20.039	30.248	33.421	52.270
		1.00	PS-12	7.5316	14.812	34.705	39.240	46.618	78.181
		1.25	PS-12	11.803	19.789	42.918	56.918	66.792	89.436
		1.50	PS-12	17.031	25.458	49.468	81.635	88.505	99.150
		2.00	PS-12	30.357	39.176	64.645	109.18	144.66	157.09
0.1	0.25	0.50	PS-12	1.0815	3.9067	5.8231	11.430	14.655	20.557
		0.75	PS-12	2.4510	6.5544	13.122	20.318	24.356	33.430
		1.00	PS-12	4.3783	9.5080	23.046	27.945	31.419	51.912
		1.25	PS-12	6.8635	12.784	30.324	37.530	44.964	66.292
		1.50	PS-12	9.9065	16.413	34.649	53.571	60.606	71.778
		2.00	PS-12	17.664	24.890	44.385	79.427	94.940	104.61
	0.50	0.50	PS-12	1.3190	4.4659	6.5431	12.322	15.886	21.903
		0.75	PS-12	2.9894	7.5134	14.742	22.089	25.995	36.231
		1.00	PS-12	5.3415	10.920	25.796	30.337	34.399	55.936
		1.25	PS-12	8.3754	14.727	33.265	42.046	49.481	70.510
		1.50	PS-12	12.091	18.986	38.287	60.143	66.711	77.287
		2.00	PS-12	21.563	29.063	49.733	86.369	106.60	116.28
	1.00	0.50	PS-12	1.8005	5.5011	7.8594	13.882	17.978	24.150
		0.75	PS-12	4.0798	9.3375	17.703	25.277	29.043	40.997
		1.00	PS-12	7.2919	13.657	30.762	34.949	39.819	63.061
		1.25	PS-12	11.438	18.549	38.894	50.337	57.721	78.387
		1.50	PS-12	16.515	24.101	45.353	72.149	77.683	87.922
		2.00	PS-12	29.464	37.481	60.258	99.825	127.85	137.64
0.2	0.25	0.50	PS-12	1.0424	3.1753	4.9393	8.8630	11.131	14.900
		0.75	PS-12	2.3605	5.7039	11.125	16.441	20.268	25.427
		1.00	PS-12	4.2171	8.5383	19.602	24.136	25.856	40.740
		1.25	PS-12	6.6127	11.690	26.782	31.850	37.346	54.247
		1.50	PS-12	9.5472	15.188	31.050	45.353	50.764	61.358
		2.00	PS-12	17.032	23.366	40.418	70.367	80.356	87.982
	0.50	0.50	PS-12	1.2577	3.5398	5.3984	9.2962	11.692	15.442
		0.75	PS-12	2.8469	6.4235	12.155	17.431	21.260	26.705
		1.00	PS-12	5.0857	9.6683	21.386	25.709	27.607	42.868
		1.25	PS-12	7.9755	13.304	28.915	34.695	40.071	57.308
		1.50	PS-12	11.516	17.377	33.795	49.504	54.588	64.871
		2.00	PS-12	20.549	27.011	44.694	75.414	87.723	95.224
	1.00	0.50	PS-12	1.6748	4.1567	6.1565	9.9815	12.573	16.262
		0.75	PS-12	3.7877	7.7128	13.853	19.065	23.007	28.715
		1.00	PS-12	6.7638	11.749	24.342	28.541	30.518	46.377
		1.25	PS-12	10.607	16.330	32.794	39.424	44.604	62.118
		1.50	PS-12	15.317	21.527	38.896	56.342	60.951	71.183
		2.00	PS-12	27.336	34.018	52.748	84.740	99.841	107.27

Nomenclature

$[B]$ – strain matrix
C – clamped edge
$[D]$ – rigidity matrix
F – free edge
k – warping factor
$[K]^e$ – element stiffness matrix
N_i – shape function for element geometry interpolation
N_{wi} – shape functions for field variable interpolation
S – simply supperted edge
w, θ_x, θ_y – field variables
v – Poisson' ratio
x, y – global coordinates
δ – tapered ratio
$\{\delta\}$ – nodal displacement vector
$\{\varepsilon\}$ – strain matrix
ρ – density of plate material
$\{\sigma\}$ – stress matrix
ϕ_x, ϕ_y – average shear strain
ω – natural frequency

References

Batoz J.L., Bathe K.J. and Ho L.W. (1980): *A study of three-node triangular plate bending elements.* – International Journal for Numerical Methods in Engineering, vol.15, pp.1771-1812.

Bhashyam G.R. and Gallagher R.H. (1984): *An approach to the inclusion of transverse shear deformation in finite element plate bending analysis*. – Computers and Structures, vol.19, pp.35-40.

Cheung Y.K. and Chen W. (1989): *Hybrid quadrilateral element based on Mindlin/Reissner plate theory*. – Computers and Structures, vol.32, pp.327-339.

Cheung Y.K. and Zhou D. (1999): *Eigen-frequencies of tapered rectangular plates with intermediate line supports*. -' International Journal of Solids and Structures, vol.36, pp.143-166.

Corr R.B. and Jennings A. (1976): *A simultaneous iteration algorithm for symmetric eigenvalue problems*. – International Journal for Numerical Methods in Engineering, vol.10, pp.647-663.

Delpak R. (1967): *Axi-symmetric vibration of shells of revolution by the finite element methods*. – M. Sc. Thesis, University of Wales, Swansea.

Ergatoudis J.G. (1966): *Quadrilateral elements in plane analysis and introduction to solid analysis*. – M. Sc. Thesis, University of Wales, Swansea.

Hughes T.J.R. and Cohen M. (1978): *The heterosis finite element for plate bending.* – Computers and Structures, vol.9, pp.445-450.

Hughes T.J.R. and Tezduyaf T.E. (1981): *Finite elements based on Mindlin plate theory with particular reference to the four-node bilinear isoparametric element*. – Journal of Applied Mechanics, vol.48, pp.587-596.

Hrabok M.M. and Hrudey T.M. (1984): *A review and catalogue of plate bending finite elements*. – Computers and Structures, vol.19, pp.479-495.

Mizusawa T. (1993): *Vibration of rectangular Mindlin plates with tapered thickness by the spline strip method.* – Computers and Structures, vol.46, pp.451-463.

Pryor C.W., Barker R.M. and Frederick D. (1970): *Finite element bending analysis of Reissner plate*. – Journal of Engineering Mechanics, ASCE, vol.96, pp.967-983.

Pulmano V.A. and Gupta R.K. (1976): *Vibration of tapered plates by finite strip method.* – ASME Journal of Engineering Mechanics, vol.102, pp.553-559.

Pugh E.D.L., Hinton E. and Zienkiewicz O.C. (1987): *A study of quadrilateral plate bending elements with reduced integration*. – International Journal for Numerical Methods in Engineering, vol.12, pp.1059-1079.

Petrolito J. (1989): *A modified ACM element for thick plate analysis*. – Computers and Structures, vol.32, pp.1303-1309.

Rao G.V., Venkataramana J. and Raj I.S. (1974): *A high precision triangular plate bending element for the analysis of thick plates*. – Nuclear. Engineering Design, vol.30, pp.408-412.

Salerno V.L. and Goldberg M.A. (1968): *Effect of shear deformations on the bending of rectangular plates*. – Journal of Applied Mechanics, ASME, vol.27, pp.54-58.

Wang C.M., Reddy J.N. and Lee K.H. (2000): *Shear deformable beams and plates*. – Elsevier Science, Amsterdam – Luisiana, New York, Oxford.

Yuan F.G. and Miller R.E. (1989): *A cubic triangular finite element for flat plate with shear*. – International Journal for Numerical Methods in Engineering, vol.28, pp.109-126.

Zienkiewicz O.C. and Taylor R.L. (1988): *The Finite Element Methods* (Two Volumes). – New York: McGraw Hill.

Zhou D. (2002): *Vibration of point supported rectangular plates with variable thickness using a set of static tapered beam functions*. – International Journal of Mechanical Science, vol.44, pp.149-164.

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