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ON PLANE WAVES IN AN ISOTROPIC LINEAR THERMOELASTIC SOLID WITH INITIAL STRESSES

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The plane wave propagation in a homogenous isotropic, thermally conducting elastic solid under normal initial stresses is studied with two thermal relaxation times. Three types of plane waves, quasi-*P*, thermal and quasi-SV waves, are shown to exist. The dependence of the velocities of these plane waves on the direction of propagation is shown graphically for different combinations of normal initial stresses.

Key words: initial stresses, wave propagation, thermoelastic, velocities.

1. Introduction

The theory of dynamic thermoelasticity is of much importance in various engineering fields such as earthquake engineering, solid dynamics, nuclear reactors, high-energy particle accelerators, etc. The theories on generalized thermoelasticity given by Lord and Shulman (1967) and Green and Lindsay (1972) have become the center of recent research due to their applications in many modern technological problems. These theories lead to further research work on wave propagation in isotropic generalized thermoelastic solids (for example, Nayfeh and Nasser (1971); Sinha and Sinha (1974); Montanro (1999); Singh (2000; 2003)).

Initial stresses are developed in the medium due to many reasons, resulting from temperature difference, quenching, creep slow process, differential external forces, gravity variations, etc. The Earth is assumed under high initial stresses. Dey *et al.* (1984; 1985) studied the propagation of waves in a medium under initial stresses. The present research note is an attempt to study the propagation of plane waves in a generalised thermoelastic solid under initial stresses with two thermal relaxation times. The numerical work is limited to Lord and Shulman theory.

2. Formulation of the problem

We consider an isotropic homogeneous thermally conducting medium with normal initial stresses S_{11} and S_{22} in two orthogonal directions *x* and *y* respectively. Following Biot (1965), Lord and Shulman (1967), Green and Lindsay (1972) and Montanaro (1999), the equations of motion in two directions under these stresses may be written as

$$B_{11}\frac{\partial^2 u}{\partial x^2} + A_3\frac{\partial^2 v}{\partial x \partial y} + A_1\frac{\partial^2 u}{\partial y^2} - \beta\frac{\partial}{\partial x}\left(T + \zeta\dot{T}\right) = \rho\frac{\partial^2 u}{\partial t^2}, \tag{2.1}$$

$$B_{22}\frac{\partial^2 v}{\partial y^2} + A_3\frac{\partial^2 u}{\partial x \partial y} + A_2\frac{\partial^2 v}{\partial x^2} - \beta\frac{\partial}{\partial y}\left(T + \zeta\dot{T}\right) = \rho\frac{\partial^2 v}{\partial t^2}, \tag{2.2}$$

$$K\nabla^2 T - \beta T_0\left[\left(\frac{\partial^2 u}{\partial x \partial t} + \tau_0\Delta\frac{\partial^3 u}{\partial x \partial t^2}\right) + \left(\frac{\partial^2 v}{\partial y \partial t} + \tau_0\Delta\frac{\partial^3 v}{\partial y \partial t^2}\right)\right] = \rho C_e\left(\dot{T} + \tau_0\ddot{T}\right) \tag{2.3}$$

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where

$$B_{11} = \lambda + 2\mu + P, \qquad B_{22} = \lambda + 2\mu, \qquad A_1 = \mu + \frac{P}{2},$$
$$A_2 = \mu - \frac{P}{2}, \qquad A_3 = \lambda + \mu + \frac{P}{2}, \qquad P = S_{22} - S_{11}, \qquad \beta = (3\lambda + 2\mu)\alpha, \tag{2.4}$$

λ, μ are Lame's constants, ρ and C_e are respectively the density and specific heat at constant strain, τ_0, τ_1 are thermal relaxation times; K is the thermal conductivity. α is the coefficient of linear thermal expansion and the dot represents time differentiation. The use of symbol Δ, in Eq.(2.3) makes these fundamental equations possible for the two different theories of the generalized thermoelasticity. For the L–S (Lord-Shulman) theory $\tau_1 = 0$, $\Delta = 1$ and for G–L (Green-Lindsay) theory $\tau_1 > 0$ and $\Delta = 0$. The thermal relaxations τ_0 and τ_1 satisfy the inequality $\tau_1 \geq \tau_0 \geq 0$ for the G–L theory only.

3. Propagation of plane waves

For a plane wave of circular frequency ω, the wave number k and phase velocity c, incident at the free boundary $y = 0$ at an angle θ with the y-axis, we may assume

$$u = X \exp(iP_1), \qquad v = Y \exp(iP_1), \qquad T = Z \exp(iP_1) \tag{3.1}$$

where X, Y, Z are amplitude factors and

$$P_1 = \omega t - k(x \sin\theta - y \cos\theta), \tag{3.2}$$

is the plane factor.

For the wave reflected at $y = 0$, we assume

$$u = X \exp(iP_2), \qquad v = Y \exp(iP_2), \qquad T = Z \exp(iP_2) \tag{3.3}$$

where
$$P_2 = \omega t - k(x \sin\theta - y \cos\theta), \tag{3.4}$$

is the phase factor associated with reflected waves. Making use of Eq.(3.1) or Eq.(3.3) in Eqs (2.1) to (2.3), we obtain

$$-(D_1 - \rho c^2)X \pm A_3 \sin\theta \cos\theta Y + \frac{i}{k}\beta\tau' \sin\theta Z = 0, \tag{3.5}$$

$$\pm A_3 \sin\theta \cos\theta X - (D_2 - \rho c^2)Y \pm \frac{i}{k}\beta\tau' \cos\theta Z = 0, \tag{3.6}$$

$$T_0 \beta\tau c^2 \sin\theta X \pm T_0 \beta\tau c^2 \cos\theta Y \frac{i}{k}(K - \rho C_e c^2 \tau^*)Z = 0 \tag{3.7}$$

where

$$D_1(\theta)=B_{11}\sin^2\theta+A_1\cos^2\theta,$$
$$D_2(\theta)=B_{22}\cos^2\theta+A_2\sin^2\theta, \tag{3.8}$$

and

$$\tau^*=\tau_0-i\omega^{-1}, \qquad \tau=\tau_0\Delta-i\omega^{-1}, \qquad \tau'=(1+i\omega\tau_1).$$

Equations (3.5) to (3.7) in *X*, *Y*, *Z* can have a nontrivial solution only if the determinant of their coefficients vanishes, i.e.

$$\zeta^3+A\zeta^2+B\zeta+C=0 \tag{3.9}$$

where

$$A=\frac{-1}{\tau^*}\left[D_1\tau^*+D_2\tau^*+D_3+\in p\right],$$

$$B=\frac{1}{\tau^*}\left[D_1D_2\tau^*+D_1D_3+D_2D_3+D_1\in p\cos^2\theta+D_2\in p\sin^2\theta+\right.$$
$$\left.-\tau^*A_3^2\sin^2\theta\cos^2\theta-2A_3\in p\sin^2\theta\cos^2\theta\right],$$

$$C=\frac{1}{\tau^*}\left[-D_1D_2D_3+D_3A_3^2\sin^2\theta\cos^2\theta\right], \tag{3.10}$$

and

$$\zeta=\rho c^2, \qquad D_3=K/C_e, \qquad \in=\frac{\beta^2T_0}{\rho C_e v_1^2}, \qquad p=\tau\tau' v_1^2, \qquad v_1^2=\frac{\lambda+2\mu+P}{\rho}. \tag{3.11}$$

The three roots ζ_1, ζ_2 of Eq.(3.9) may be obtained by using Cardan's method.

It may be noted that whether we take the upper sign or lower sign in Eqs (3.5) to (3.7), we get the same three values of ζ by Eq.(3.9). These roots give the analytical expressions for the velocities of propagation of quasi–*P*, thermal and quasi–SV waves respectively. Therefore, in a two-dimensional generalized thermoelastic solid with initial stress, there exists three plane waves whose phase velocities depend on the direction of propagation, frequency (ω) and normal initial stress.

4. Special cases

(i) For an isotropic and homogeneous medium under normal initial stresses

$$\beta=0, \qquad K=0\Rightarrow D_3=0, \qquad \in=0.$$

The cubic Eq.(3.9) reduces to a quadratic equation which gives the expressions of velocities of longitudinal $\left(C_L^2\right)$ and transverse waves $\left(C_T^2\right)$ as obtained by Dey *et al.* (1984) in an isotropic homogeneous medium under normal initial stresses.

(ii) For an isotropic and homogeneous medium

$$P = 0, \qquad \beta = 0, \qquad K = 0 \Rightarrow D_3 = 0, \qquad \in = 0.$$

Equation (3.9) reduces to a quadratic equation, which gives the expressions for velocities of propagation of *P* and SV waves in a two-dimensional model of isotropic elastic media.

5. Numerical analysis

We restrict our study for the case of Lord and Shulman theory only. From Eq.(3.9), we can find the square of phase velocities of quasi–*P*, thermal and quasi-SV waves as $c_1^2 = \zeta_1/\rho$, $c_2^2 = \zeta_2/\rho$ and $c_3^2 = \zeta_3/\rho$ respectively. The numerical values of $c_1^2/(\mu/\rho)$, $c_2^2/(\mu/\rho)$ and $c_3^2/(\mu/\rho)$ are computed for different combinations of normal initial stress. We introduce normal initial stress parameters as

$$\eta_1 = \frac{S_{11}}{2\mu}, \qquad \eta_2 = \frac{S_{22}}{2\mu}.$$

The following parameters in SI units are also used for numerical computations

$$\lambda/\mu = 1, \qquad \tau_0 = 0.05s, \qquad \tau_1 = 1s, \qquad \tau_0 = 293^{\mathbf{0}} K, \qquad \rho = 2300\, kg/m^3,$$

$$\in = 0.053, \qquad K = 5.19 \times 10^2\, J/ms^{\mathbf{0}}K, \qquad C_e = 1.6235 \times 10^4\, J/kg^{\mathbf{0}}K.$$

Figures 1 to 3 show a comparison between the velocity curve for cases involving different combinations of biaxial initial stresses and the case when the medium is free of initial stresses. The quasi–*P* waves are represented by curves shown in Fig.1, when $\omega = 5$. Deviation of values of velocities of longitudinal waves for $\eta_1 = -0.4$ and $\eta_2 = 0.8$ (curve 1) from the initial stress free case (curve 2) is considerable for the range $0 < \theta \le 90^{\mathbf{0}}$. The deviation for the case when $\eta_1 = 0.8$ and $\eta_2 = 0.4$ (curve 3) is also significant.

The quasi–*P* is affected due to thermal disturbances. If we neglect thermal disturbances, curves 1, 2, 3 reduce to curves 4, 5 and 6 respectively. The thermal waves are represented by curves in Fig.2, when $\omega = 5$. The deviation of $c_2^2/(\mu/\rho)$ for $\eta_1 = -0.4$, $\eta_2 = 0.8$ and for $\eta_1 = 0.8$, $\eta_2 = 0.4$ from the initial stress free case (curve 1) are shown by curves 2 and 3 respectively.

The quasi–SV waves are represented by curves shown in Fig.3, when $\omega = 5$. Deviations of values of velocities of quasi-SV waves for $\eta_1 = -0.4$, $\eta_2 = 0.8$ (curve 2) and $\eta_1 = 0.8$, $\eta_2 = 0.4$ (curve 3) from the initial stress free case (curve 1) are considerable. The SV wave remains unaffected by thermal disturbances. The curves 4, 5 and 6 show same variations as curves 1, 2 and 3 respectively.

It may be pointed here that the above numerical analysis fairly agrees with those of Dey *et al.* (1984).

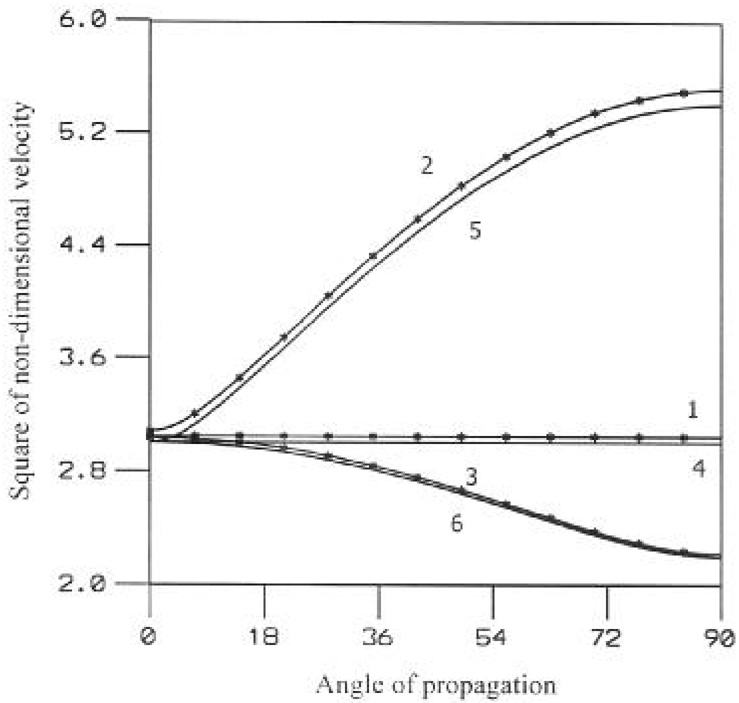

Fig.1. Variations of square of non-dimensional velocity $c_1^2/(\mu/\rho)$ of quasi-*P* wave with angle of propagation for different combinations of η_1 and η_2.

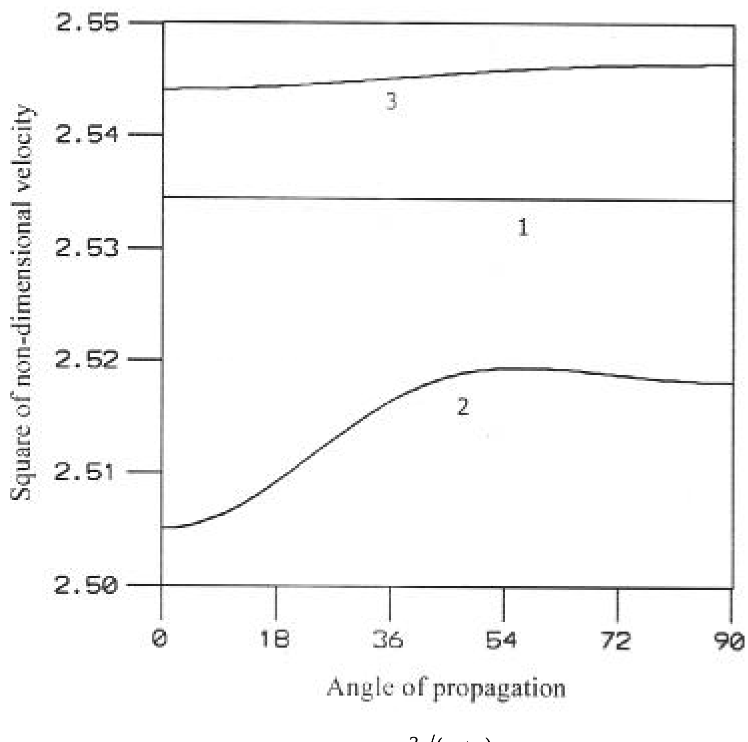

Fig.2. Variations of square of non-dimensional velocity $c_2^2/(\mu/\rho)$ of thermal wave with angle of propagation for different combinations of η_1 and η_2.

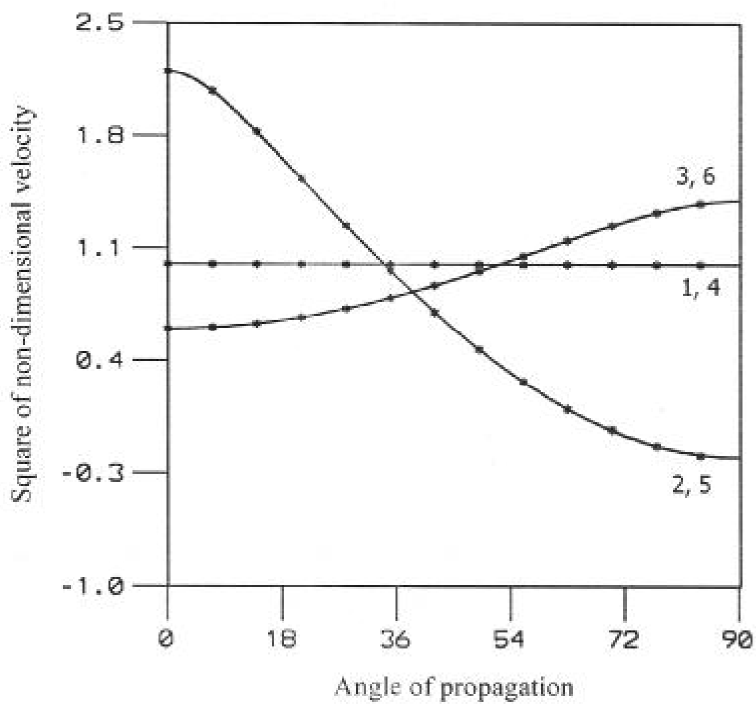

Fig.3. Variations of square of non-dimensional velocity $c_3^2/(\mu/\rho)$ of quasi-SV with angle of propagation for different combinations of η_1 and η_2.

Nomenclature

c – phase velocity
C_e – specific heat at constant strain
k – wave number
K – thermal conductivity
S_{11}, S_{22} – normal initial stresses
T_0 – uniform temperature
α – coefficient of linear thermal expansion
λ, μ – Lame's constants
ρ – density of medium
τ_0, τ_1 – thermal relaxation times
ω – circular frequency
$\in$ – thermo-coupling coefficient

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